The Frenemy Paradox

The Emergence of Instability with the
Advent of Modern Social Relations

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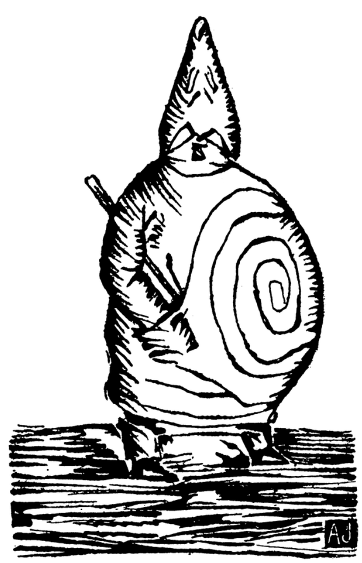
Abstract

In this analysis we interrogate the social implications of the newly codified relationship of "frenemies." In particular, we will show that the existence of even a single frenemy relation in a social network leads to manifestly paradoxical outcomes. We conclude by addressing the implications of these findings for social media outlets such as Facebook and society at large.



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1 Introduction

In recent years, with the advent of the age of social media, sociologists and information theorists, among others, have been privy to one of the largest collections of social data ever collected. The capacity to, for the first time, build social models capable of being experimental tested and verified have led to a revolution of our understanding of human relations at an unprecedented level of mathematical rigor and elegance. Nevertheless, the formal developments in this emerging field have been lacking, with need for codified definitions and careful analysis of first principles. In this page, we hope to do just this.

We will begin by defining the notions of both friendship and enemyship using a set theoretic language. We will extend this theory to include the time evolution of such relationships, attempting to classify a wide class of modifiers. These definitions will be informed by a series of powerful theorems, the most important of which is the long overlooked Theorem of Arthashastra. These notions will then be presented in terms of social networks and graph theory to show our Frenemy Paradox. Finally, we will interpret these results in the settings of the social network Facebook and general society, posing a series of rather disturbing conjectures.

2 Friendship

Let us begin by understanding friendship.

Formal work here begins the discovery of the celebrated Friendship Paradox of Scott L. Feld in 1991 [1], wherein a simple use of the arithmetic-geometric mean the Cauchy-Schwarz inequalities led to the a peculiar result. We restate it here:

> Let a social network be represented by an undirected graph $G = (V, E)$, where the set of vertices V corresponds to the people in the social network, and the set of edges E correspond to the friendship relations between people. The number of friends of a person in the social network is defined the degree d of a particular vertex $v \in V$, given by $d(v)$. It follows[1] that total average degree of a friend is strictly greater than the average degree of a random vertex.

Here, we must be care to note that Feld assumed that friendship is a symmetric relation, that is, if, say, Alice is friends with Bob then Bob is necessarily friends with Alice. We will continue with this assumption, though point the reader to [2] and [3] for a canonical review of work relaxing this condition.

We begin with a set of definitions.

Definition 1. A **social network** V is a finite set where element $v \in V$ are called **people**.

Definition 2. A social network V is equipped with a symmetric relation $\Leftrightarrow$, where, for elements $v, w \in V$, if $v \Leftrightarrow w$, we say v is **friends** with w.

A lay-person might scoff at such an emaciated attempt to categorify a rather nuanced human relation, but to such individuals we respectively turn our esteemed backs.

2.1 Friendship Coefficient

The above definition, as noted by Goodman et al. [4], fails to account for the natural metric imposed on the continuum of friendship. A thorough analysis requires that we extend the relation to a weighted relation, wherein we introduce the Friendship Coefficient to indicate the "level of Friendship." In preparation to definition, we introduce a series of preliminary concepts.

Definition 3. The **Similarity Index** is a map $S : V \times V \to \mathbb{R}^+$, where, for two people v and w, we assign a value $S(v, w)$. We write $S(v, v) = \infty$, where ∞ is to be understood as the point of $\mathbb{R}^+$ at infinity.

Definition 4. The **Cluster Index** is a map $C : V \times V \to \mathbb{R}^+$, where if $v \Leftrightarrow w$, it is defined as

$$C(v, w) = \frac{1}{|(d(w) \cap d(v)) \backslash \{v, w\}|} \sum_{n \in (d(w) \cap d(v)) \backslash \{v,w\}} S(v, n).$$

and zero otherwise. Here we write $d(w)$ for the set of friends of $d(w)$, and we do not sum over $S(v, v)$ or $S(v, w)$ to avoid over counting.

Heuristically, the Similarity Index accounts for compatibility between any two people, where a high Similarity Index number between two people indicates that they have a lot of interests

[1] albeit non-trivially.

and beliefs in common. Note that we have bounded this value from below to allow us to use it as a coefficient in the growth of friendship. Finally, we assume that any person will be complete similarity with themselves.

Likewise, the Cluster Index is a weighted average over the Similarity Indices of mutual friends, attempting to account for the principle that friends cluster together. It is important to realize that $C(v, w)$ is implicitly a function of time, as the number $d(w)$ can increase or decrease over time.

Definition 5. The **Friendship Coefficient** is a map $\mathcal{C}_F : V \times V \times \mathbb{R} \to \mathbb{R}_{\geq 0}$, defined by

$$\mathcal{C}_F(v, w, t) = \left[C(v, w) + \int_{t_0}^{t} S(v, w) \left| e^{i\omega t} \right| dt \right] \Theta(t - t_0),$$

where t_0 is the time at which v and w first held the relation $v \Leftrightarrow w$ and ω is the **Frequency of Hang**. Here $\Theta(x)$ is the Heaviside step-function, which is equal to zero for negative values, and one for positive values. We say that $v \Leftrightarrow w$ at a time t if and only if $\mathcal{C}_F(v, w, t) \neq 0$.

Note that the Friendship Coefficients allow us to map each friends to a real number, and thus establish a well-ordered relation of friendship, which allows us to call a particular friendship (v, w) greater than[2] (v', w') at a given time t, if and only if $\mathcal{C}_F(v, w, t) > \mathcal{C}_F(v', w', t)$.

Further, the Friendship Coefficient indicates that friendship will build over time. We introduce an oscillatory growth with ω, to reflect the experimental fact that friends that hangout more frequently become better friends. For a in depth review of this phenomenon we refer to [5]. We will discuss the implications of this the next section.

2.2 Evolution of Friendship

An important insight of the definition of the Friendship Coefficient is the inclusion of a time variable. In preparation of our analysis of social network, we examine the evolution of a simple network.

Let us denote each element $v \in V$ by a node, where connections between two points v and w indicate that $v \Leftrightarrow w$ and the thickness of the connection is a function of $\mathcal{C}_F(v, w, t)$. Consider the simple social network given below.

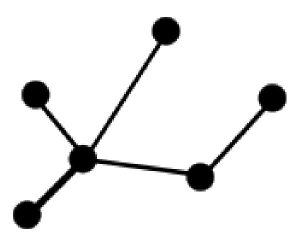


Figure 1: A simple network of friendship, where the set of people has order six.

[2]or 'stronger" in the colloquial lexicon.

We will now make an important, but very reasonable assumption, which we shall call the **Transitivity of Friendship**: There exists a time $t > t'$, such that if $\mathcal{C}_F(v, w, t'), \mathcal{C}_F(w, u, t') > 0$, then we have $\mathcal{C}_F(v, u, t) > 0$. Again, falling back to our lay speech, this indicates that friends of friends will eventually become friends. Thus, at a finite latter time, the network of Figure 1, will approach the network below.

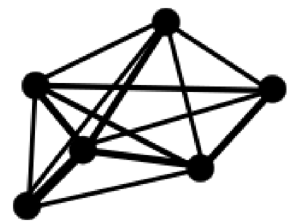


Figure 2: The time evolved graph of Figure 1.

Thus, given the assumptions of our model, we see that in a finite time[3], for any elements $v, w \in V$, if V forms a connected graph, then $v \Leftrightarrow w$. This is a very strong result, and we will discover that it leads to a plethora of problems.

3 Enemyship

Classically, Friendship has been defined to included negative values, where zero indicates strangers, and negative values refer to enemies. Here we will discard this construction, working in a paradigm where the Enemyship space is orthogonal to the Friendship space. In direct analogy to Friendship, we have the basic definition below.

Definition 6. A social network V is equipped with a symmetric relation $\bowtie$, where, for elements $v, w \in V$, if $v \bowtie w$, we say v is **enemies** with w.

In fact, the majority of our machinery from Section 2 carries into this definition, where we can replace the Similarity Index C by the **Difference Index** D, and, as we shall prove, there is a analogue to the Cluster Index.

3.1 Arthashastra's Theorem

Recall the classic result of Arthashastra [6], where in

> The king who is situated anywhere immediately on the circumference of the conqueror's territory is termed the enemy.
> The king who is likewise situated close to the enemy, but separated from the conqueror only by the enemy, is termed the friend (of the conqueror).

In the modern language, we have **Arthashastra's Theorem** given by:

[3] Note that such times may exceed the estimated life of the universe, let along the life of any given element $v \in V$.

Theorem 1. *For any $v, w, u \in V$, if $v \bowtie w$ and $w \bowtie u$, then $v \Leftrightarrow u$.*

Proof. Easy. □

In lay-language, this is simply "the enemy of my enemy is my friend."

3.2 Corrections to the Friendship Coefficient

Arthashastra's Theorem gives us a correction to the Friendship Coefficients, and we have the following adapted definition.

Definition 7. The **Corrected Friendship Coefficient** is a map $\mathcal{C}_{AF} : V \times V \times \mathbb{R} \to \mathbb{R}_{\geq 0}$, defined by

$$\mathcal{C}_{AF}(v, w, t) = \left[C(v, w) + E(v, w) + \int_{t_0}^{t} S(v, w) \left| e^{i\omega t} \right| dt \right] \Theta(t - t_0),$$

where t_0 is the time at which v and w first held the relation $v \Leftrightarrow w$, ω is the Frequency of Hang, and $E(v, w)$ is the number of elements $u \in V$ such that $u \bowtie w$ and $u \bowtie v$.

3.3 Enemyship Coefficients

As noted above, we may introduce a set of definitions in complete analogy to our Friendship space to the Enemyship space. At risk of repeating ourself, we define the Enemy coefficient as follows.

Definition 8. The **Enemyship Coefficient** is a map $\mathcal{C}_E : V \times V \times \mathbb{R} \to \mathbb{R}_{\geq 0}$, defined by

$$\mathcal{C}_E(v, w, t) = \left[F(v, w) + \int_{t_0}^{t} D(v, w) \left| e^{i\hat{\omega} t} \right| dt \right] \Theta(t - t_0),$$

where t_0 is the time at which v and w first held the relation $v \bowtie w$, $\hat{\omega}$ is the **Frequency of Hate**, $D(v, w)$ is the beforestated Difference Index, and $F(v, w)$ is the weighted number of elements $u \in V$ such that $u \bowtie v$ and $u \Leftrightarrow w$, or $u \bowtie w$ and $u \Leftrightarrow v$, where the weight of the respective nonzero Corrected Friendship Coefficient. We say $v \bowtie w$ at a time t if and only if $\mathcal{C}_E(v, w, t) \neq 0$.

Here $F(v, w)$ is to be understood as the effect of Arthashastra's Theorem. We are immediately presented with a corollary.

Corollary 1. *For any $v, w, u \in V$, if $v \Leftrightarrow w$ and $w \bowtie u$, then there exists some time t_0, such that after a time t_0, we have $v \bowtie u$.*

Proof. Since $v \Leftrightarrow w$, we have $\mathcal{C}_{AF}(v, w) > 0$. Note that w satisfies the conditions of the function $F(v, u)$ and thus we have $F(v, u) > 0$. By the definition of $\mathcal{C}_E(v, u, t)$, we see that if $t > t_0$, we have $v \bowtie u$. Given our assumption of ergodicity of a connected social landscape, namely that after a finite time, any element of V will acquire a relation with any other connected element of V, we find the result follows. □

This corollary should be interpreted to mean that "the friend of your enemy is your enemy" as well as "the enemy of your friend is your enemy."

Similar to our discussion of the time evolution of friendship, let us denote the graphic relation of enemy by a dotted line between two elements $v, w \in V$ with again the line thickness as the value of $\mathcal{C}_E$. Consider the initial network on the left of Figure 3 below. It should be a clear result of corollary 1 that after a finite time, we arrive at the network in the middle. After another finite interval, we arrive at the terminal diagram on the right.

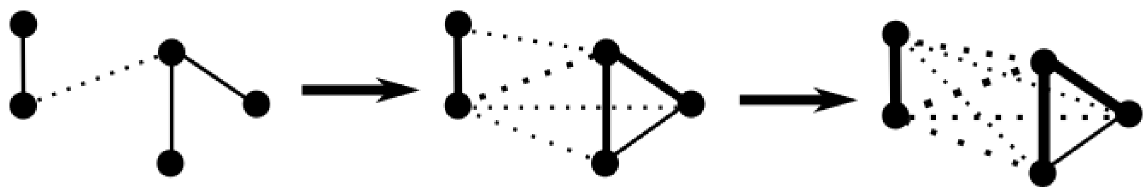


Figure 3: Three timestep evolution of a initial social network, the intermediary time, and the terminal time.

Note here that in the terminal graph, every person has a relation with every other person. In fact, we have

Proposition 1. *Given any connected social network V, after a finite time, for any $v, w \in V$, we have $v \Leftrightarrow w$ or $v \bowtie w$.*

Proof. This easily follows from Corollary 1 and our assumption concerning the evolution of Friendship. □

4 Frenemy

The term "frenemy," a simultaneous oxymoron and portmanteau (of "friend" and "enemy") emerged in the early 1950s and have since shook the world [7]. The very idea that a member of a social network could hold both relations of friend and enemy at the same time may strike may of our readers as blasphemy, but rest assured, a new generation of relations are indeed emerging, and require our attention.

Our definition is brief, and if the reader has been attentive in our previous discussion, the motivation should be clear.

Definition 9. For $v, w \in V$, if $v \Leftrightarrow w$ and $v \bowtie w$, we say that v and w are **frenemies**.

To prepare for what follows, we denote the relation of frenemy on a social network graph as below.

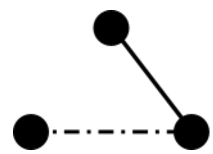
Figure 4: Frenemies

5 The Paradox

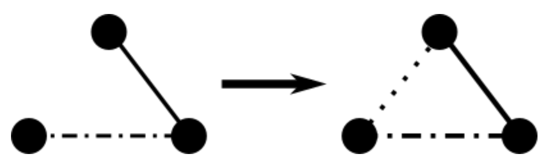
We are now prepared to reveal the **Frenemy Paradox**.

Theorem 2. *If any connected social network V contains people that are frenemies, then in a finite time, all people in the social network will be frenemies.*

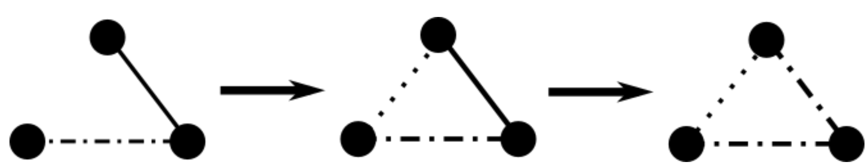
Proof. Let us begin by consider a simple network $V = \{v, w, u\}$ and assume that $v \Leftrightarrow w$ and $v \bowtie w$. Since V must be connected, without loss of generality, let us assume that u is connected to w. We have two cases: either $u \Leftrightarrow w$ or $u \bowtie w$.
In the first case, we have the graph given below



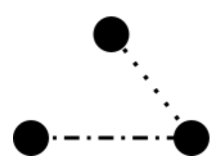
Now, using Corollary 1, after finite time, say t_1, we have $u \bowtie v$. The evolution goes as



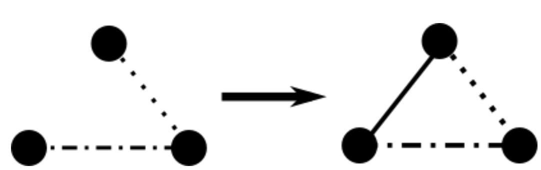
Now, running Corollary 1 again, but using the fact that $w \Leftrightarrow v$ and $v \bowtie u$, we find that at a later time t_2, we have $w \bowtie u$. Hence, we find that u and w become frenemies, as shown below.



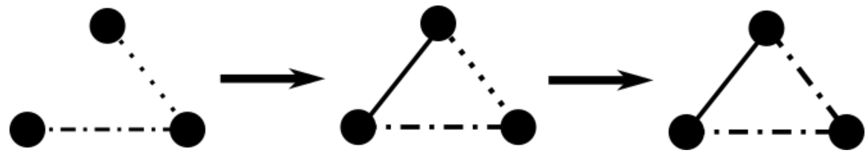
Turning to our second case, suppose that $u \bowtie w$. We then have the graph



Using Arthashastra's Theorem, we find that in a finite time t'_1, that $u \Leftrightarrow v$ as shown below.



Finally, using the assumption of Transitivity of Friendship, we find that in a latter time t'_2, since $v \Leftrightarrow w$ and $v \Leftrightarrow u$, we have $u \Leftrightarrow w$. Hence, we have



Therefore, given either a Friendship or a Enemyship relation to a member of a Frenemy relation will evolve to a Frenemy relation in a finite amount of time. Hence, with the assumption of a connected social network, we see that the theorem follows. □

Therefore we arrive at the peculiar result:

> The existence of even a single frenemy will lead to every person simultaneously being everyone's friend and everyone's enemy.

6 Implications

The Frenemy Paradox is far reaching and concerning. Clearly the very knowledge of the term "frenemy" implies their existence, and thus we are certainly already on the road to the "frenemy singularity." Nevertheless, there may be hope.

In our model, we have been keen to note that this evolution occurs in an unspecified "finite time." According to the Poincaré recurrence theorem [8], the universe will also repeat itself in a "finite amount of time," but the savvy physicist will note that the expected amount of time is many orders of magnitude greater than the age of the universe. Does the same apply to the Frenemy Paradox? Let us consider two cases.

6.1 Facebook

The majority of collected social network information comes to us from the gold standard of social media, Facebook. To analyze the implications of the Frenemy Paradox for this case, we must first determine how one defines enemies and friends in this system.

It is manifestly clear that the Facebook notion of friend is encoded in the status of being "friends" on Facebook[4]. It is far less clear how one determines the status of enemy. Clearly, many enemies exist on Facebook that are not friends, but we are interested in the occurrence of a Frenemy. After careful consideration, it dawns on any focused thought that a "Facebook Frenemy" is a friend that has been unfollowed. Unfortunately, this is not a symmetric relation, though we contest that the dearth of "likes" from a friend who has unfollowed you will lead to the sentiment of frenemy.

[4] We took the time to reconsider our career choices after writing this sentence.

Since the second author has recently unfollowed the first author, there exists at least one "Facebook Frenemy." In an appeal to the well-known "Degrees of Separation" theorems, we may consider Facebook as a connected social network. Therefore the Frenemy Paradox applies.

How are we then to interpret the finite time in the statement of our paradox? Clearly, Facebook allows for a near instant evolution of relations, and thus it seems likely that we cannot appeal to the argument a la Poincaré recurrence. This leads us to the following conjecture.

Conjecture 1. *As a result of the Frenemy Paradox, in the next decade, every member of Facebook will have both friended and unfollowed every other member of Facebook. This was the original intent of creator MZ, allowing for the creation of an entire ad-space where each isolated node holds all the user information of the total network..*

6.2 Society

It is the belief of these authors that everyone is already the enemy and the friend of everyone they know. We leave the proof as an exercise to the reader.

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