TWISTED SUPERGRAVITY, AND FAMILY DONALDSON INVARIANTS

by

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ABSTRACT OF THE DISSERTATION

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At its heart, this dissertation investigates the relationship between two of physics' most important symmetries, diffeomorphisms and gauge transformations. Directed by the study of the metric dependence of solutions to the instanton equation of Yang-Mills theory on a smooth four manifold $\mathbb{X}$, we construct a model of equivariant cohomology of the space of gauge connections $\mathcal{A}$ and metrics $\mathsf{Met}(\mathbb{X})$ with respect to the semi-direct product of gauge transformations $\mathcal{G}$ and diffeomorphisms $\mathsf{Diff}_+(\mathbb{X})$. Generalizing topologically twisted $\mathcal{N} = 2$ super Yang Mills theory, we use our model to present a new set of transformation laws and action which allow for the construction of new diffeomorphism invariants of $\mathbb{X}$ associated to families of metrics. These are, conjecturally, the fabled family Donaldson invariants. Surprisingly, we also identify our model as a subsector of $\mathcal{N} = 2$ twisted supergravity on a background with only certain components of the gravitino activated. In addition, we provide perspective on future directions for these developments.

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A third and unbridled barrage I next send to my interlocutors A.B., D.B., R.G., J.Y., A.K., and NicoDemon. We each and all spent many a nights together in mutual revelry, trial, and confusion.

To my father, whose shadow provided shade when the world became too much to bare and who taught me the power of helping others. To my mother, whose light provided guidance far beyond the nest and who taught me to love myself. To T., whose force provided inspiration and who taught me the importance of duty. I love you all.

I must also thank my cat, Nótt, whose contribution to this work had to be edited down to a simple “sdwdskbvlfvlvwkhvflhqfh.”

And finally, an undeserved nod to pataphysicians everywhere.

Contents

Introduction

Physics is the study of change. Typically, changes are measured through rates along a world line, leading to our familiar notions of velocity, acceleration, jerks, snap, crackle, pop, and all that. Mathematically, these rates require us to take a series of derivatives with respect to a time variable along the world line. On a manifold, the proper space to allow for a curved or topologically non-trivial universe, infinite iterations of derivatives in any combination of directions require the transition functions between charts to be likewise infinitely differentiable. This leads us to consider *smooth* manifolds as candidates for the universe of a physical theory.

Naively, given a manifold $\mathbb{X}$ where the transition functions between charts are merely continuous, one might expect that, if available, a choice of smooth structure would be unique up to diffeomorphism. Shockingly, this is not always the case. Indeed, in 1956, John W. Milnor constructed a smooth structure on the seven sphere $\mathbb{S}^7$ which was homeomorphic, but not diffeomorphic to the standard Euclidean 7-sphere [67]. It was further shown that there were a total of 28 non-equivalent smooth structures on the homotopic $\mathbb{S}^7$ [48]. Such smooth manifolds which have a smooth structure different from the standard are called *exotic.* In dimensions less than four, there are no exotic manifolds and in dimensions greater than four, there are at most a finite number . In four dimensions, where we seemingly live, things get wild.

For $n \neq 4$, the manifold $\mathbb{R}^n$ has exactly one smooth structure, namely the one specified by the traditional flat Euclidean metric [79]. In an incredible twist to the story, for $\mathbb{R}^4$ there are an uncountable number of exotic smooth structures [8, 39, 66]. Further, research into the possible smooth structures on closed, simply-connected, smoothable manifolds seems to suggest that the presence of exotic structures is the norm, not the exception [75]. For the case of the four sphere $\mathbb{S}^4$, this question, namely, "Does an exotic four sphere exist?" is the final open piece of the generalized Poincaré conjecture, the so called "last man standing" among the problems of classical geometric topology [33]. Indeed, in stark contrast to every other dimension, there is no known classification of smooth structures for even a single smoothable four manifold.

Despite the unruliness of four manifolds, some features have been tamed. In fact, it is nearly settled as to which simply-connected closed four manifolds allow for a smooth structure [23, 36, 41].[1] This progress was in large part due to the incredible work of Simon K. Donaldson, when, in 1983, he introduced the eponymous Donaldson polynomial invariants. Letting $\mathbb{X}$ be a closed, oriented, smooth four manifold, we can write the generating function for the polynomials of $\mathbb{X}$ as

$$\mathsf{Z}_{\mathrm{D}}[g, p, s] = \sum_{\ell, r \geq 0} \frac{p^{\ell}}{\ell!} \frac{s^{r}}{r!} \mathfrak{P}_{\mathrm{D}}^{\ell, r}, \tag{0.1}$$

where g is a metric on $\mathbb{X}$ and p and s are formal variables associated to a point and surface in $\mathbb{X}$ respectively, thus, for a fixed degree, defining a polynomial on $H_0(\mathbb{X}) \oplus H_2(\mathbb{X})$. Here the $\mathfrak{P}_{\mathrm{D}}^{\ell, r}$ are rational numbers which are independent of the metric on $\mathbb{X}$, so long as there is a sufficiently large vector space of self-dual two form, that is, for $b_2^+(\mathbb{X}) > 2$. Hence, if computed, these invariants provide a potential method of distinguishing between different smooth structures.

But what is it that the Donaldson polynomials are actually computing? Before we answer this question, let us take a step back to understand *topological invariants* in general. If two manifolds can be deformed into each other, the invariants of each manifold will be the same, as is often parlayed into the image of a confused topologicist unsure at breakfast whether to drink out of a donut or take a bite out of his coffee cup; to them, the two objects are indistinguishable on account of having only one

[1] We make this statement precise in Section 0.1.2 below.

hole. Following this intuition, the topological invariants of a manifold should count something, such as holes. Even the dimension of a manifold is a topological invariant, as, no matter how hard one tries, no one has or will ever be able to deform a manifold into a higher dimension. If one leaves their lower dimensional manifolds behind and makes it into these higher dimensions, you'll find higher dimensional holes, such as the one inside of a sphere. Unfortunately just counting holes is insufficient to distinguish two smooth manifolds i.e. the number of holes is not a *complete* invariant. Indeed, both $\mathbb{S}^7$ and an exotic $\mathbb{S}^7$ each have only one "seven dimensional hole," but they are not the same smooth manifolds.

Thankfully there are myriad other topological invariants. Some are $\mathbb{Z}_2$-valued and only give a "yes" or a "no," such as orientability, which measures a manifold's ability to have a well-defined notion of handedness. On an un-orientable surface, such as the Möbius strip, a left-handed person is just a right-handed person who needs to go for a walk. Similarly, a theory of physics on such a surface would not allow for chiral particles, which leads us to a result: Because we find chiral particles in our universe, we must live on a oriented manifold. Here, an observation in a physical theory has just told us something about the world it lives in.

Extrapolating this approach, we are led to ask, what can a physical theory teach us about the topology of the underlying spacetime? The answer, it turns out, is precisely the one we initially sought. One of our more sophisticated theories is the Yang-Mills theory, which is the overarching field theory of both quantum electrodynamics and quantum chromodynamics and forms the basis of our understanding of the Standard Model. The classical solutions to its equations of motion generalize the classical Maxwell equations and its quantum field theory gives all the beautiful machinery of the strong and electroweak forces of nature. Quantum mechanically, there is another type of solution, namely, a *instanton*. Instantons are topologically non-trivial field configurations that are localized in both space and time and minimize the action of the Yang-Mills theory . This is to be juxtaposed with the notion of a particle which is only localized in space, and, as an object in spacetime, is a worldline. In constrast, to an observer, instantons are blips that last for only an instant. Moreover, Wick rotating over to Euclidean space, the number of possible instanton solutions depends on the manifold $\mathbb{X}$. This leads us to our answer, the Donaldson polynomial invariants are, for a fixed number of instantons, counting the number of ways these instantons can all be put on the manifold $\mathbb{X}$!

Donaldson's work was motivated by physics, but surely our physical theory of Yang-Mills on a manifold $\mathbb{X}$ will depend on the metric? How can we build a theory which is "topological" and ensure that when we ask the physics to count the instanton solutions it won't get stuck on a choice of metric? In 1988, Edward Witten, in a deft maneuver guided by the intuition of a physicist, discovered a method to "twist" the fields of the four dimensional $\mathcal{N}=2$ supersymmetric Yang Mills theory with gauge group G and arrived at the notion of a cohomologically topological field theory [85]. This theory is equipped with a differential $\mathcal{Q}$ that squares to zero on all gauge invariant functionals of the fields. With this differential, we can write action of the theory $S_{\rm UV}$ as

$$S_{\rm UV} = \mathcal{Q}(V_{\rm UV}) + \frac{i\tau_0}{16\pi}\int_{\mathbb{X}} {\rm Tr} F_A \wedge F_A, \tag{0.2}$$

where $V_{\rm UV}$ is a functional of the twisted vector multiplet fields, F_A is Yang Mills field strength, and τ_0 is the complex coupling constant. Note that this is the sum of a $\mathcal{Q}$-exact term and a topological term. Further, the stress energy tensor $T_{\mu\nu}$, defined as

$$\delta_g S_{\rm UV} = \frac{1}{2}\int_{\mathbb{X}} d^4x\sqrt{g}\delta g^{\mu\nu} T^{\rm UV}_{\mu\nu}, \tag{0.3}$$

for an infinitesimal change $g^{\mu\nu} \longrightarrow g^{\mu\nu} + \delta g^{\mu\nu}$, can be written as

$$T^{\mathrm{UV}}_{\mu\nu} = \mathcal{Q}\Lambda^{\mathrm{UV}}_{\mu\nu}. \tag{0.4}$$

We can then define the partition function of this theory as

$$\mathsf{Z}_{\mathrm{W}}[g] = \int [d\mathrm{VM}] e^{-S_{\mathrm{UV}}}. \tag{0.5}$$

where we have conducted the path integral over the twisted vector multiplet fields. Noting that $\mathcal{Q}$ originates from the supersymmetry of the untwisted theory, we see that[2]

$$\begin{aligned} \delta_g \mathsf{Z}_{\mathrm{W}}[g] &= \int [d\mathrm{VM}] \left(\frac{1}{2} \int_{\mathbb{X}} d^4x \sqrt{g} \delta g^{\mu\nu} T^{\mathrm{UV}}_{\mu\nu} \right) e^{-S_{\mathrm{UV}}}, \\ &= \int [d\mathrm{VM}] \mathcal{Q} \left(\frac{1}{2} \int_{\mathbb{X}} d^4x \sqrt{g} \delta g^{\mu\nu} \Lambda^{\mathrm{UV}}_{\mu\nu} e^{-S_{\mathrm{UV}}} \right) = 0. \end{aligned} \tag{0.6}$$

Therefore, the partition function is formally independent of the choice of metric on $\mathbb{X}$, that is, the theory is topological!

Moreover, the theory is equipped with $\mathcal{Q}$-closed, gauge invariant observables. On a simply-connected, closed manifold, we have two observables of note, the 0-observables $\mathcal{O}^{(0)}(p_0)$ associated to a point $p_0 \in \mathbb{X}$ and the 2-observables $\mathcal{O}^{(2)}(\Sigma)$ associated to a closed surface $\Sigma \subset \mathbb{X}$. These choices, p_0 and Σ, only depends on the homology class, up to $\mathcal{Q}$-exact terms, which themselves decouple from the theory [52]. We can write

[2]We point the eager reader to (0.118)-(0.123) for a more careful treatment of this important result.

the expectation value of the generating function of these observables as

$$\begin{aligned}\mathsf{Z}_W[g,p,s] &= \int [d\mathrm{VM}] e^{p\mathcal{O}^{(0)}(p_0)+s\mathcal{O}^{(2)}(\Sigma)} e^{-S_{\mathrm{UV}}} \\ &= \sum_{\ell,r\geq 0} \frac{p^\ell}{\ell!}\frac{s^r}{r!} \int [d\mathrm{VM}] \left(\mathcal{O}^{(0)}(p_0)\right)^\ell \left(\mathcal{O}^{(2)}(\Sigma)\right)^r e^{-S_{\mathrm{UV}}}.\end{aligned} \tag{0.7}$$

The beauty of this theory is that, up to an overall constant prefactor, we have

$$\mathsf{Z}_{\mathrm{D}}[g,p,s] = \mathsf{Z}_{\mathrm{W}}[g,p,s]. \tag{0.8}$$

Henceforth, we shall then only refer to the unified *Donaldson-Witten partition function* Z_{DW}.

Were this the only benefit of the physics interpretation, one would be excused for marvelling and moving on, but there's more! In 1994, together with Nathan Seiberg, Witten provided an exact low energy effective action for the untwisted $\mathcal{N}=2$ supersymmetric Yang Mills theory for SU(2) [76, 77], allowing for approximate computations of correlation functions. Here, the physics is mapped from the high energy UV theory to the low energy IR theory, where the dynamics are comprised of fluctuations about the quantum vacua. In the IR, the gauge symmetry breaks to U(1), though at certain points in the moduli space of quantum vacua, massless particles enter the story. Turning to the twisted story, Z_{DW} is independent of the metric for choices of $\mathbb{X}$ with $b_2^+ > 1$, and thus one can scale to long length scales, or, correspondingly, low energy limits. Since the twisted theory is topological, this mapping is not just an approximation, but an exact correspondence.

Led by the tools of physicists, we then arrive at the outcome that the Donaldson-Witten invariants can be computed as an integral over the moduli space of quantum vacua for the U(1) twisted Seiberg-Witten theory. Such vacua are parameterized by the expectation value of the $\mathcal{O}^{(0)}(p_0)$ observable of the theory, which is denoted u. Further, in 1997, Gregory W. Moore, with Witten, conducted the explicit integral over the *u-plane* of these vacua, leading to the relation

$$\mathsf{Z}_{\mathrm{DW}} = \sum_{u=\pm 1} \mathsf{Z}_{\mathrm{SW}}^{u} + \mathsf{Z}_{u}, \tag{0.9}$$

where Z_{SW} are contributions from the points on the u-plane where there are monopole solutions and Z_u is the so-called u-plane integral [71]. For $b_2^+(\mathbb{X}) > 1$, they showed that $\mathsf{Z}_u = 0$, and thus the Donaldson-Witten invariants can be written entirely in terms of monopole solutions, greatly simplifying their computation. Meanwhile, at $b_2^+(\mathbb{X}) = 1$, each of the above terms is only piecewise constant over the space of metrics, leading to the phenomenon of *wall crossing,* where the values jump across certain domain walls.

This is only the tip of the iceberg, as after this work of exchanging the messy "non-abelianness" of Donaldson theory for the tractable abelian nature of Seiberg-Witten theory, many of the old theorems of four manifold theory were given new, simpler proofs and the connection further inspired entirely new directions of study [25, 70]. This exchange between mathematicians and physicists is rightly heralded as one of the greatest success stories in the burgeoning field of *physical mathematics* [5, 68] This field is characterized by its use of the techniques and intuitions of quantum field theory and its cousins to pose and prove concrete problems in mathematics. The goal of this thesis is to continue this tradition.

An early remark by Donaldson [24] saw fit to consider a further generalization of his invariants, namely to generalize them to higher degree forms on the classifying space of orientation-preserving diffeomorphisms, $\mathsf{BDiff}_+(\mathbb{X})$. To understand this, consider Z_{DW} as function over the space of metrics on $\mathbb{X}$, denoted $\mathsf{Met}(\mathbb{X})$, which is, for $b_2^+(\mathbb{X}) > 1$, invariant under orientation preserving diffeomorphisms. Hence, putting aside for the moment issues of singularities, we have Z_{DW} as a function on the space $\mathsf{Met}(\mathbb{X})/\mathsf{Diff}_+(\mathbb{X})$, which contains the same topological data as $\mathrm{BDiff}_+(\mathbb{X})$. This immediately introduces the notion of considering higher degree differential forms on $\mathsf{Met}(\mathbb{X})/\mathsf{Diff}_+(\mathbb{X})$, and, along with an appropriate differential, the cohomology $H^*(\mathsf{Met}(\mathbb{X})/\mathsf{Diff}_+(\mathbb{X}))$. Then, integrating such an n-forms over an n-paramter family of metrics in $\mathsf{Met}(\mathbb{X})$ would yield *family Donaldson invariants.*

Returning to those issues of singularities in the quotient space "$\mathsf{Met}(\mathbb{X})/\mathsf{Diff}_+(\mathbb{X})$," we are brought to the mathematical theory of *equivariant cohomology,* which is designed precisely to tackle the cohomology of manifolds with a (not necessarily free) group action. As we will explore in detail, the original densities of the Donaldson-Witten invariants are themselves elements of equivariant cohomology on the space of gauge connections $\mathcal{A}$ with respect to the group of gauge transformations $\mathcal{G}$, that is, elements of $H_{\mathcal{G}}(\mathcal{A})$, where $\mathcal{Q}$ plays the role of the differential[3]. In order to explore our family Donaldson invariants, we must extend this model of equivariant cohomology to the product space

$$\mathbb{M} = \mathcal{A} \times \mathsf{Met}(\mathbb{X}) \tag{0.10}$$

with the action of the group

$$\mathbb{G} = \mathcal{G} \rtimes \mathsf{Diff}_+(\mathbb{X}). \tag{0.11}$$

Then, denoting the differential for the cohomology $H_{\mathbb{G}}(\mathbb{M})$ by $\mathbb{Q}$,[4] we will extend the original $\mathcal{Q}$-closed action S_{UV} to a $\mathbb{Q}$-closed action $\mathbb{S}_{\mathrm{UV}}$. The generating function for

[3]With the addition of two modules over $H_{\mathcal{G}}(\mathcal{A})$, or *contractible pairs,* which we will discuss in due time.

[4]For obvious reasons, will never refer to the rational numbers as $\mathbb{Q}$ in this work.

our family Donaldson invariants is then defined as

$$\mathsf{Z}[g,\Psi,\Phi] = \int [d\mathrm{VM}] e^{-\mathbb{S}_{\mathrm{UV}}}. \tag{0.12}$$

Here, $g \in \mathsf{Met}(\mathbb{X})$, $\Psi \in \Omega^1(\mathsf{Met}(\mathbb{X}))$, and $\Phi \in \mathrm{Vect}(\mathbb{X})$ are the generating fields for the complex of $H_{\mathsf{Diff}_+(\mathbb{X})}(\mathsf{Met}(\mathbb{X}))$, which is equipped with the differential d. From this perspective, integrating over the vector multiplet fields is equivalent to projecting down from our total complex of the $H_{\mathbb{G}}(\mathbb{M})$ model to the base complex of $H_{\mathsf{Diff}_+(\mathbb{X})}(\mathsf{Met}(\mathbb{X}))$. Since $\mathbb{Q}\mathbb{S}_{\mathrm{UV}} = 0$, a simple chain map argument reveals that

$$\mathsf{dZ}[g,\Psi,\Phi] = 0, \tag{0.13}$$

Moreover, expanding our generating function in degrees, we have

$$\mathsf{Z}[g,\Psi,\Phi] = \sum_{m=0}^{\infty} \mathsf{Z}^{[m]} \tag{0.14}$$

and due to the fact that d is homogeneous of degree one, we find

$$\mathsf{dZ}^{[n]} = 0, \tag{0.15}$$

for all n. Here, each $\mathsf{Z}^{[n]}$ is degree n element of $H_{\mathsf{Diff}_+(\mathbb{X})}(\mathsf{Met}(\mathbb{X}))$, and thus we can integrate it over an n-parameter family of metrics to obtain the fabled family Donaldson invariants. Of course, by construction $\mathsf{Z}^{[0]} = \mathsf{Z}_{\mathrm{DW}}[g]$, where the right hand side is the original Donaldson-Witten invariants without the insertion of observables.

Surprisingly, we can also obtain these invariants from a physically motivated direction. To do so, consider four dimensional $\mathcal{N} = 2$ Euclidean supergravity, as in [17]. In joint work with Moore, Roček, and Saxena to appear [12], we will show that one is able to first twist and truncate this theory and then by restricting to a symmetric gravitino background arrive at a consistent theory of twisted supergravity. Moreover, the fields and the transformations laws of this theory exactly coincide with those of our model of $H_{\mathbb{G}}(\mathbb{M})$ and its modules! Further, using the *chiral density formula* and *superconformal tensor calculus* in supergravity, one obtains an action $\mathbb{S}^{\mathrm{t}}_{\mathrm{UV}}$ which is equal to $\mathbb{S}_{\mathrm{UV}}$ up to a $\mathbb{Q}$-exact term!

Hence, herein we hope to open the door to an exciting new chapter in the continued dialogue between physics and mathematics in our quest to understand four manifolds. With this view ahead, we also provide our initial investigations into the inclusion of family observables. Finally, we also provide a IR formulation of this entire story, so that future work might repeat the analysis of Moore and Witten in [71] and understand the expected wall-crossing for these invariants on manifolds with $b_2^+ > 1$.

The structure of this work is as follows. In order to generalize, we must first settle that which is to be extended, so we begin with Section 0, where we conduct a crash course in basic four manifold theory, then move to a likewise concise review of $\mathcal{N} = 2$ supersymmetry in four dimensions. We conclude the preliminaries by conducting Witten's twist and reconstruct the Donaldson-Witten invariants, with an emphasis on equivariant cohomology and the Mathai-Quillen formalism. Section 1 is the heart of the paper, wherein we carefully construct the Cartan model of equivariant cohomology for $H_{\mathbb{G}}(\mathbb{M})$, include two modules, and then, in a lengthy excursus, show that this model is equivalent to truncated twisted supergravity on a symmetric gravitino background. Section 2 presents our generalized construction of both a UV action $\mathbb{S}_{\mathrm{UV}}$ and a IR action $\mathbb{S}_{\mathrm{IR}}$. We further take another excursus into twisted supergravity, and present its construction of the action and then show that it is equal to our own up to a $\mathbb{Q}$-exact term. Section 3 presents the hero of our story, the family invariants. After some light exploration in their features, we then extend our gaze to the horizon, where in Section 4 we present preliminary investigations into the inclusion of observables, the computation of our invariants, and the prospects of wall-crossing phenomenon. We then conclude.

We will assume familiarity with certain aspects of differential topology, algebraic topology, as well as quantum field theory.[5] For the truly uninitiated, we point to the beautiful book by Alexandru Scorpan [75] for a mathematical approach and the likewise wonderful (and shorter!) text of Jose Labastida and Marcos Mariño [53] for a more physical approach. We would also be unduly remiss to not mention the expert reviews [10] and [70].

[5] To the level that, math-wise, every item in the bulleted list below is understood, and physics-wise, the discussion in the introduction about Seiberg-Witten theory was understood. We will try to be as curt as possible without truly losing any reader who is at the position the author was when they began their graduate studies.

0 Foundations

Our story features a cast of main characters. In the leading role, we always take $\mathbb{X}$ to be a closed, oriented, smooth four manifold. In addition, we hold here for reference, the other main players:

- G : Gauge group, i.e. a compact semisimple Lie group.
- $\mathfrak{g}$: Lie algebra of G.
- P : Principal G-bundle $P \to \mathbb{X}$.
- $\mathcal{A}$: Space of G-connections on P.
- $\mathcal{G}$: Group of gauge transformations, i.e. the group of fibre preserving automorphisms of P.
- $\mathsf{Lie}\,\mathcal{G}$: Lie algebra of $\mathcal{G}$.
- $\mathsf{Met}(\mathbb{X})$: Space of Riemannian metrics on $\mathbb{X}$.
- $\mathsf{Diff}_+(\mathbb{X})$: Lie group of orientation-preserving diffeomorphisms of $\mathbb{X}$.
- $\mathsf{diff}(\mathbb{X})$: Lie algebra of $\mathsf{Diff}_+(\mathbb{X})$.
- $\mathbb{M} = \mathcal{A} \times \mathsf{Met}(\mathbb{X})$.[6]
- $\mathbb{G} = \mathcal{G} \rtimes \mathsf{Diff}_+(\mathbb{X})$.

[6] For ease of notation and not to be confused with the Monster group.

0.1 Four Manifold Theory

0.1.1 Basics

A *topological four manifold* is a topological space[7] X which is locally $\mathbb{R}^4$, that is, for every point $p_0 \in X$ there is an open neighborhood $U \subset X$ of p_0 and a homeomorphism (a continuous map which has a continous inverse) $\varphi_U : U \longrightarrow \varphi_U(U) \subset \mathbb{R}^4$. These maps are called *charts*, though often the definition is extended to refer to the domains as well, and often written (U, φ_U). Given two non-disjoint neighborhoods U and V, we have a homeomorphism from $\varphi_U(U \cap V)$ to $\varphi_V(U \cap V)$ given by $\varphi_V \circ \varphi_U^{-1}$, which we call the *transition function.* We consider two topological manifolds to be equivalent as topological manifolds if there exists a homeomorphism between them. Further, we say a topological four manifold is *closed* if it both compact and has no boundary. Finally, a closed topological four manifold is said to be *orientable* if its top homology $H_4(X, \mathbb{Z})$ is isomorphic to the integers $\mathbb{Z}$. Here a choice of generator is an *orientation*, which gives one an *oriented* manifold. We restrict our entire discussion to connected, closed, oriented manifolds.

A *smooth four manifold* is a topological four manifold with a collection of charts so that every transition function is smooth (differentiable to all orders). We say a topological manifold X is *smoothable* if there is a selection of its charts which cover X such that every transition function is smooth. We call such a selection a *smooth structure.* We consider two smooth manifolds to have equivalent smooth structure is there exists a diffeomorphism (a smooth function which has a smooth inverse) between them.[8] Note that these definitions allow for two smooth manifolds X and Y to be equivalent as topological manifolds, but different as smooth manifolds.

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[7]Which needs to be both separable and Hausdorff, which to a physicists need only mean that it isn't a pathological nightmare.

[8]Note that a smooth homeomorphism need not be a diffeomorphism. For example consider $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = x^3$, whose inverse is not differentiable at 0.

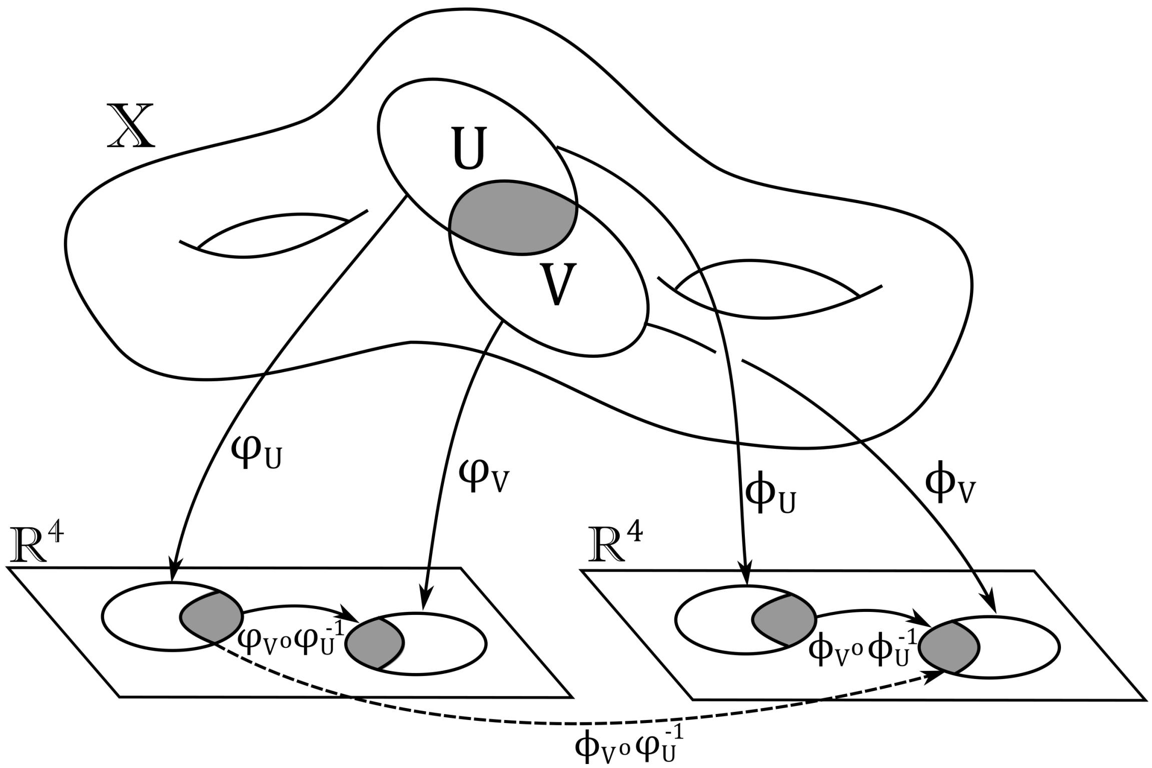

Figure 1: If $\{\varphi\}$ and $\{\phi\}$ are two independent collections of charts such that all transition functions $\{\varphi_V \circ \varphi_U^{-1}\}$ and $\{\phi_V \circ \phi_U^{-1}\}$ are smooth, then they are both individually smooth structures on $\mathbb{X}$. If every $\{\phi_V \circ \varphi_U^{-1}\}$, as in the dotted map, is smooth, then the two smooth structures are equivalent. If a single one is not, then the two smooth structures are different.

Following these definitions, one is immediately presented with the question of classification. The question comes in three parts, namely

- **First**: What is the classification of topological four manifolds?
- **Second**: What are the conditions for a topological four manifold to be smoothable?
- **Third**: What is the classification of smooth structures on smoothable four manifolds?

The first question is, in general intractable, as given any finitely presented group

H, one can construct a topological four manifold X with its fundamental group $\pi_1(X) = H$. The question of distinguishing between two finitely presented groups is known as the word problem, and it's unfortunately undecidable [74]. So much for generality. Thankfully, once one restricts to simply-connected topological four manifolds, the classification has been settled. Thus we turn to the intersection form.

0.1.2 Intersection Form

The most basic topological invariants of any manifold are its homology groups, $H_n(X, \mathbb{Z})$, and cohomology groups $H^n(X, \mathbb{Z})$. They are abelian groups and the rank of the nth homology group is called the *nth Betti number* of X, denoted by b_i. Moreover, when X is simply-connected, both its homology and cohomology groups are free, so at this point we all but turn our backs to non-simply-connected manifolds.

We can then define the *intersection form* Q_X as the symmetric bilinear form

$$Q_X : H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \longrightarrow \mathbb{Z}, \tag{0.1}$$

defined as the cup product between 2-cocycles. Unfortunately, cup products are confusing, so let's recall Poincaré duality. Since the manifolds we consider are oriented, we have the canonical isomorphism

$$H_n(X, \mathbb{Z}) \cong H^{4-n}(X, \mathbb{Z}). \tag{0.2}$$

In four dimensions, this gives us an isomorphism between $H^2(X, \mathbb{Z})$ and $H_2(X, \mathbb{Z})$, which leds to a geometric picture of Q_X. If S_α and S_β in $H_2(X, \mathbb{Z})$ are representatives duals to 2-cocycles α and β, then the intersection form can defined as the signed intersection number of the surfaces S_α and S_β,

$$Q_X(\alpha, \beta) = S_\alpha \cdot S_\beta. \tag{0.3}$$

In another salute to Poincaré duality, Q_X is unimodular, that is, it has $\det(Q_X) = 1$. Therefore Q_X is a symmetric integral unimodular bilinear form. It is important to stress that we are working over the integers, and thus a change of basis is an element of $\mathrm{GL}_{b_2}(X, \mathbb{Z})$, not $\mathrm{GL}_{b_2}(X, \mathbb{R})$. For this reason, Q_X is an interesting invariant.

The intersection form itself has a number of algebraic invariants. First, its rank is simply $b_2(X)$. Next, we can diagonalize the matrix Q_X over the real numbers $\mathbb{R}$ and count the number of positive and number eigenvalues. We call them b_2^+ and b_2^- respectively. The *signature* of Q_X is then defined as

$$\text{sign } Q_X = b_2^+ - b_2^-. \tag{0.4}$$

We say Q_X is *definite* if either b_2^+ or b_2^- are zero, and *indefinite* otherwise. Lastly, we have the *parity*, which is said to be *even* if, for all classes $\alpha \in H^2(X, \mathbb{Z})$, $Q_X(\alpha, \alpha) \in 2\mathbb{Z}$, and *odd* otherwise.

Turning to the classification of possible intersection forms, there has been wonderful success for the indefinite case. Here we have *Serre's Classification Theorem,* which tells us that two indefinite integral symmetric bilinear unimodular forms are isomorphic if they have the same rank, signature, and parity [78]. Concretely, this means that if Q_X is indefinite and odd, then it is isomorphic to

$$m[+1] \oplus n[-1]. \tag{0.5}$$

and if it is even, then it is isomorphic to

$$\pm m\mathrm{E}_8 \oplus nH, \tag{0.6}$$

where $m, n \in \mathbb{Z}^+$ and

$$\mathrm{E}_8 = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{bmatrix}, \quad \text{and} \quad H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{0.7}$$

Definite forms on the other hand are far more complicated. In the even case, *van der Blij's lemma* indicates that the rank must be a multiple of 8 [83]. At rank 8, we have just E_8, two at rank 16, the 24 Niemeier lattices at rank 24, but then suddenly more than eighty million at rank 32. The landscape is even less clear on the odd side. Thankfully, for every fixed rank, there is only a finite number of integral symmetric bilinear unimodular forms up to isomorphism.

Having some semblance of algebraically-possible intersection forms, one asks which ones are realized as the intersection forms for topological four manifolds. The answer, it turns out, is all of them. More so we have *Freedman's Classification Theorem* [32, 34]. It states that, for any integral symmetric unimodular form Q, there is a closed simply-connected topological four manifold that has Q as its intersection form. This further divides into two cases:

- If Q is even, then there is exactly one topological four manifold up to homeomorphism.
- If Q is odd, then there are exactly two such topological manifolds up to homeomorphism, at least one of which is not smoothable.

Hence, we have answered the first of our original three questions. Let us now turn to smooth topological four manifolds.

0.1.3 Donaldson Invariants

Let $\mathbb{X}$ be a closed, oriented, smooth four manifold. We will also continue to restrict to the simply-connected case. With a smooth structure in hand, we can now speak of the tangent bundle $T\mathbb{X}$, the cotangent bundle $T^*\mathbb{X}$, and the space of differential forms $\Omega^*(\mathbb{X})$. Further, with the exterior derivative d, we can speak of the de Rham cohomology $H^*_{\mathrm{DR}}(\mathbb{X}, \mathbb{R})$ of $\mathbb{X}$, which is taken over the real number $\mathbb{R}$. In our case, where there is no torsion in $H^*(\mathbb{X}, \mathbb{Z})$, we will view $H^2_{\mathrm{DR}}(\mathbb{X}, \mathbb{Z})$ as an integral lattice inside of $H^2(\mathbb{X}, \mathbb{R})$. With this perspective, we can write the intersection form on two 2-cocycles, represented by two-forms, as

$$Q_{\mathbb{X}}(\alpha, \beta) = \int_{\mathbb{X}} \alpha \wedge \beta. \tag{0.8}$$

A fundamental result of differential geometry is that every smooth manifold has a *Riemannian metric* g [55]. Here, g assigns to each point $p \in \mathbb{X}$ a positive-definite inner product on the tangent space $T_p\mathbb{X}$. Note that a single smooth manifold $\mathbb{X}$ can have many different metrics, and we denote the space of all possible ones, namely the moduli space of metrics on $\mathbb{X}$, by $\mathsf{Met}(\mathbb{X})$. This space is topologically uninteresting on its own right, since it is contractible.[9]

[9] To see this, take any two $g_0, g_1 \in \mathsf{Met}(\mathbb{X})$ and consider the path $g_n = ng_0 + (1-n)g_1$ for

With a metric (and an orientation), we can define the *Hodge star operator* $\star$. In four dimensions, this is a map $\star : \Omega^n(\mathbb{X}) \longrightarrow \Omega^{4-n}(\mathbb{X})$ which squares to one, hence has eigenvalues ± 1. In our conventions, for $\omega \in \Omega^2(\mathbb{X})$, in a coordinate basis, we can write

$$\star\omega_{\mu\nu} = \frac{1}{2}\sqrt{g}\epsilon_{\mu\nu\rho\sigma}g^{\rho\rho'}g^{\sigma\sigma'}\omega_{\rho'\sigma'}, \tag{0.9}$$

where we write $\sqrt{g}$ for the square-root of the determinant of g. Our Levi-Cevita symbol holds no metric dependence, but we can see the Hodge star's explicit dependence on the metric. Since $\star$ is an involution on $\Omega^2(\mathbb{X})$ there is a splitting into *self-dual* two-forms $\Omega^+(\mathbb{X})$ with eigenvalues $+1$ and *anti-self-dual* two forms $\Omega^-(\mathbb{X})$ with eigenvalues -1. Given any two-form ω, we will use an index $\pm$ to denote its self-dual or anti-self-dual parts respectively. Thus we have

$$\omega^{\pm}_{\mu\nu} = \frac{1}{2}\omega_{\mu\nu} \pm \frac{1}{4}\sqrt{g}\epsilon_{\mu\nu\rho\sigma}g^{\rho\rho'}g^{\sigma\sigma'}\omega_{\rho'\sigma'}. \tag{0.10}$$

This splitting descends in de Rham cohomology to a splitting of $H^2_+(\mathbb{X},\mathbb{R})$ and $H^2_-(\mathbb{X},\mathbb{R})$. Thus, considering the definition of integration over differential forms, we have

$$Q_{\mathbb{X}}(\omega^+,\omega^+) = \int_{\mathbb{X}} \omega^+ \wedge \omega^+ = \frac{1}{2}\int_{\mathbb{X}} d^4x\sqrt{g}\omega^+_{\mu\nu}\omega^{\mu\nu}_+ > 0, \tag{0.11}$$

and

$$Q_{\mathbb{X}}(\omega^-,\omega^-) = \int_{\mathbb{X}} \omega^- \wedge \omega^- = -\frac{1}{2}\int_{\mathbb{X}} d^4x\sqrt{g}\omega^-_{\mu\nu}\omega^{\mu\nu}_- < 0. \tag{0.12}$$

Further, $\omega^+ \wedge \omega^- = 0$. Hence, we see that $H^2_+(\mathbb{X},\mathbb{R})$ is a maximal positive definite subspace for $Q_{\mathbb{X}}$ and $H^2_-(\mathbb{X},\mathbb{R})$ is a maximal negative definite subspace for $Q_{\mathbb{X}}$. This

$n \in [0,1]$. Since each g_n is still a positive-definite inner product, it readily follows that $\mathsf{Met}(\mathbb{X})$ can be contracted down to a single point.

means that we have the identification

$$b_2^+(\mathbb{X}) = \dim\, H^2_+(\mathbb{X},\mathbb{R}) \quad \text{and} \quad b_2^-(\mathbb{X}) = \dim\, H^2_-(\mathbb{X},\mathbb{R}). \tag{0.13}$$

With a smooth structure, we can also delve into *gauge theory*, which studies *gauge connections* on various types of fibre bundles, in particular principal bundles.[10]Let G be our *gauge group*, namely a compact Lie group (typically $G = \mathrm{SU}(2)$ or $\mathrm{SO}(3)$) with Lie algebra $\mathfrak{g}$. We then take $P \to \mathbb{X}$ to be a principal G-bundle, with a space of G-connections $\mathcal{A}(P)$. Here $\mathcal{A}$ is an affine space, where the difference of two of its elements is in $\Omega^1(\mathbb{X}, \mathsf{ad}\, P)$, that is, a one form on $\mathbb{X}$ with values in the adjoint bundle. Given a connection $A \in \mathcal{A}(P)$, we can define a gauge covariant derivative[11]

$$D_A = d + A, \tag{0.14}$$

and compute its curvature, or, in physics parleance, its field strength, as

$$F_A = dA + A \wedge A, \tag{0.15}$$

and, in local coordinates, as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{0.16}$$

Here, the brackets are the Lie brackets of $\mathfrak{g}$. In addition, we have the group of fibre preserving automorphisms of P, which we call the *group of gauge transformations* $\mathcal{G}$.

[10]For in depth reviews, specifically in four dimensions, we point the reader to the texts [26, 30, 35].

[11]Mathematicians will often reserve the term connection for the differential D_A and call A a local connection 1-form. This is done since A transforms inhomogenously under gauge transformations, while D_A transforms covariantly. We follow the nomenclature of physicists.

For $\epsilon \in \mathsf{Lie}\,\mathcal{G}$, we have the left-action on the connection and curvature as

$$\delta_\epsilon A = [\epsilon, A] - d\epsilon \quad \text{and} \quad \delta_\epsilon F_A = [\epsilon, F_A]. \tag{0.17}$$

We can fully classify principal SU(2)-bundles by their second Chern class $k = c_2(P) \in \mathbb{Z}$. Using Chern-Weil theory, this integer can be written as the integral

$$k = -\frac{1}{16\pi^2}\int_{\mathbb{X}} \mathrm{Tr} F_A \wedge F_A, \tag{0.18}$$

where the trace is normalized so that each possible integer k can be realized. We shall refer to the number k as the *instanton number.* On the other hand, one can also consider the integral

$$S_{\mathrm{YM}} = \frac{1}{2}\int_{\mathbb{X}} \mathrm{Tr} F_A \wedge \star F_A = \frac{1}{4}\int_{\mathbb{X}} d^4x\sqrt{g}\,\mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}], \tag{0.19}$$

which we call the *Yang-Mills functional* or *Yang-Mills action.* Note, thanks to the trace, that both S_{YM} and k are of course invariant under the action of $\mathcal{G}$.

It is natural to consider the minimization of S_{YM}. To do so, note that we can write

$$S_{\mathrm{YM}} = \frac{1}{2}\int_{\mathbb{X}} \left[\mathrm{Tr} F_A^+ \wedge F_A^+ - \mathrm{Tr} F_A^- \wedge F_A^-\right], \tag{0.20}$$

and also

$$k = -\frac{1}{16\pi^2}\int_{\mathbb{X}} \left[\mathrm{Tr} F_A^- \wedge F_A^- + \mathrm{Tr} F_A^+ \wedge F_A^+\right]. \tag{0.21}$$

Hence, for $k = 0$, S_{YM} has absolute minima on flat connections $F_A = 0$, for $k < 0$, on connections which satisfy $F_A^- = 0$, and for $k > 0$, on connections with $F_A^+ = 0$. The difference between the cases of $k < 0$ and $k > 0$ is simply due to a choice of orientation, so we restrict our attention to $k > 0$. Thus we are interested in solutions to

$$F_A^+ = 0, \tag{0.22}$$

which is knwon as the *instanton equation.* Note that $F_A^+ = 0$ implies that $\star F_A = -F_A$, so that solutions are the so-called anti-self-dual connections.[12] Further, we only care about connections up to equivalence under gauge transformations, so we need only consider equivalence classes of connections $[A] \in \mathcal{A}/\mathcal{G}$. Thus, the space of interest is the *instanton moduli space*

$$\mathcal{M}_{k,g} = \left\{ [A] \in \mathcal{A}/\mathcal{G} \,\middle|\, F_A^+ = 0 \right\}, \tag{0.23}$$

where we specify a fixed metric g, due to the Hodge star's dependence on a metric. $\mathcal{M}_{k,g}$ is, in general, non-compact, and due to the quotient by $\mathcal{G}$, also singular. Thankfully, for $b_2^+ > 0$ and a generic metric, the moduli space for $G = \mathrm{SU}(2)$ is a finite-dimensional orientable smooth manifold of dimension

$$\dim \mathcal{M}_{k,g} = 8k - 3(1 - b_1 + b_2^+). \tag{0.24}$$

Working more carefully, with the singularities at the forefront, let us define the trivial bundle $\mathcal{E}_g \longrightarrow \mathcal{A}$ with total space

$$\mathcal{E}_g = \mathcal{A} \times \Omega_g^{2,+}(\mathbb{X}, \mathsf{ad}\, P). \tag{0.25}$$

We then define a section s of $\mathcal{E}_g$

$$s(A) = F_A^+. \tag{0.26}$$

[12] Connections are locally one-forms, so cannot be themselves anti-self-dual. When we call a connection anti-self-dual, we are commenting on the properties of its curvature.

It is clear then that solutions to the instanton equation are the zero locus of s, namely $s^{-1}(0)$. Further, in order to only consider solutions up to gauge transformations, have need to quotient $s^{-1}(0)$ by $\mathcal{G}$. Unfortunately for us, there are connections in $\mathcal{A}$ that are fixed under some gauge transformations, so the resulting quotient will have singularities at these points. Ignoring the center of the gauge group, when a particular connection is fixed by any elements of $\mathcal{G}$, we say it is *reducible*, and *irreducible* otherwise. For the case of SU(2), a reducible connection will have an isotropy group isomorphic to U(1), and its adjoint bundle will split into a direct sum of U(1) bundles. To a physicist, this is the phenomenon of the gauge symmetry breaking.

Inspecting the definition $\delta_\epsilon A$ in (0.17) and D_A in (0.14), we can identify reducible connections by whether or not there exists $\epsilon \in \mathsf{Lie}\,\mathcal{G}$ that satisfy

$$\delta_\epsilon A = -D_A \epsilon = 0. \tag{0.27}$$

Locally on $\mathcal{A}$, we can understand D_A here as a map

$$D_A : \mathsf{Lie}\,\mathcal{G} \longrightarrow T_A\mathcal{A}, \tag{0.28}$$

so that a connection is irreducible if and only if it D_A has a trivial kernel, $\mathrm{Ker}\, D_A = 0$. Likewise, the image of D_A can be considered as the gauge orbit of the connection A. We have a $\mathcal{G}$ invariant metric on $\mathcal{A}$, so we can further define the adjoint of this operator

$$D_A^\dagger : T_A\mathcal{A} \longrightarrow \mathsf{Lie}\,\mathcal{G}. \tag{0.29}$$

It then follows that the tangent space at a point $A \in \mathcal{A}$ will decompose into its gauge orbit and the kernel of $D_A^\dagger$, so we write

$$T_A\mathcal{A} = \mathrm{Im}\, D_A \oplus \mathrm{Ker}\, D_A^\dagger. \tag{0.30}$$

Together, this means the neighborhood of a irreducible connection in $\mathcal{A}/\mathcal{G}$ will be modeled by $\mathrm{Ker}\, D_A^\dagger$, while the one for a reducible connection will be $\mathrm{Ker}\, D_A^\dagger$ quotiented by the isotropy group of elements $\epsilon \in \mathcal{G}$ with $D_A\epsilon = 0$.

Next, we want to bring the section $s(A)$ back into the story. Suppose, we have a irreducible connection A which satisfies $s(A) = F_A^+ = 0$. Deforming this solution by $\psi \in \Omega^1(\mathbb{X}, \mathsf{ad}\, P)$ (since $\mathcal{A}$ is an affine space), in order to remain on the zero locus of $s(A)$ we require that $F_{a+\psi}^+ = 0$. Hence, to first order, we have

$$(D_A\psi)^+ = 0. \tag{0.31}$$

The left hand side of this condition can be expressed by the map[13]

$$\boldsymbol{\nabla} s : T_A\mathcal{A} \longrightarrow \Omega_g^{2,+}(\mathbb{X}, \mathsf{ad}\, P) \tag{0.32}$$

so that tangent vectors in $\mathrm{Ker}\, \boldsymbol{\nabla} s$ are precisely those that maintain the instanton equation. Since the gauge orbits of $\mathcal{A}$ were given by $\mathrm{Im}\, D_A$ we then conclude that for a representative irreducible connection $[A] \in \mathcal{A}/\mathcal{G}$, we have

$$T_{[A]}\mathcal{M}_{k,g} = \mathrm{Ker}\, \boldsymbol{\nabla} s \cap \mathrm{Ker}\, D_A^\dagger. \tag{0.33}$$

[13]We must be careful not to conflate the symbol $\boldsymbol{\nabla}$ here with the metric covariant derivative ∇ which will appear latter. We would have written D_A^+, but this leads to some rather perverse equations when it meets with $D_A^\dagger$.

This space can be realized as the kernel of the map

$$\mathbb{F} = \boldsymbol{\nabla} s \oplus D_A^\dagger : T_A\mathcal{A} \longrightarrow \mathsf{Lie}\,\mathcal{G} \oplus \Omega_g^{2,+}(\mathbb{X}, \mathsf{ad}\,P). \tag{0.34}$$

All of this technology can be condensed into the *Atiyah-Hitchin-Singer complex* [2]

$$0 \longrightarrow \mathsf{Lie}\,\mathcal{G} \xrightarrow{D_A} T_{\mathcal{A}} \xrightarrow{\boldsymbol{\nabla} s} \Omega_g^{2,+}(\mathbb{X}, \mathsf{ad}\,P) \longrightarrow \mathrm{Coker}\,\boldsymbol{\nabla} s \longrightarrow 0, \tag{0.35}$$

where the cokernel is defined as $\mathrm{Coker}\,\boldsymbol{\nabla} s = \Omega_g^{2,+}(\mathbb{X}, \mathsf{ad}\,P)/\mathrm{Im}\,\boldsymbol{\nabla} s$. In our work, it is generically the case that $\mathrm{Coker}\,\boldsymbol{\nabla} s = 0$.[14] This is a chain complex,[15] as we compute

$$(\boldsymbol{\nabla} s \circ D_A)\epsilon = [F_A^+, \epsilon] = 0, \tag{0.36}$$

since $[A] \in \mathcal{M}_{k,g}$, where $F_A^+ = 0$. Taking the homology groups of the complex, we have its index given by

$$\mathrm{Index}_{\mathrm{AHS}} = -\dim H^0 + \dim H^1 - \dim H^2, \tag{0.37}$$

where $H^0 = \mathrm{Ker}\,D_A$, $H^1 = \mathrm{Ker}\,\boldsymbol{\nabla} s/\mathrm{Im}\,D_A$, and $H^2 = \mathrm{Coker}\,\boldsymbol{\nabla} s$. Since, in this case $\dim \mathrm{Coker}\,D_A^\dagger = \dim \mathrm{Ker}\,D_A$, we have

$$\begin{aligned} \mathrm{Index}(\mathbb{F}) &= \dim \mathrm{Ker}\,\mathbb{F} - \dim \mathrm{Coker}\,\mathbb{F} \\ &= \dim \mathrm{Ker}\,\boldsymbol{\nabla} s + \dim \mathrm{Ker}\,D_A^\dagger - \dim \mathrm{Coker}\,\boldsymbol{\nabla} s - \dim \mathrm{Coker}\,D_A^\dagger \\ &= \dim(\mathrm{Ker}\,\boldsymbol{\nabla} s/\mathrm{Im}\,D_A) - \dim \mathrm{Coker}\,\boldsymbol{\nabla} s - \dim \mathrm{Ker}\,D_A = \mathrm{Index}_{\mathrm{AHS}} \end{aligned} \tag{0.38}$$

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[14] We will use this word "generically," quite often. Technically, it means "on all but a set of measure zero," and intuitively it means "pretty much always."

[15] This is to say that the composition of two consecutive maps in the diagram is always zero.

This index is also called the *virtual dimension* of the moduli space for the following reason. When $[A]$ is irreducible (and Coker $\nabla s = 0$), this is exactly the dimension of the $T_{[A]}\mathcal{M}_{k,g}$ and hence the dimension of $\mathcal{M}_{k,g}$ in (0.24)! On the other hand, when there are reducible connections, $H^0 \neq 0$ and the index decreases, as does the dimension of $T_{[A]}\mathcal{M}_{k,g}$, which is exactly what we expect at singularities. Fortunately for the work here, we will rarely talk about reducible connections and thus moving forward we write $\mathcal{A}/\mathcal{G}$ with the nuances therein understood.

Having settled some of various objects that can be built on smooth manifolds, we are ready to return to our second main question, namely which topological four manifolds are smoothable. In 1982, following investigations into the nature of the instanton moduli space, Donaldson provided a partial answer in his eponymous *Donaldson's Theorem* [23]. It states that if a topological four manifold $\mathbb{X}$ with a definite intersection form $Q_{\mathbb{X}}$ is smoothable then it must be the case that

$$Q_{\mathbb{X}} = m[+1] \quad \text{or} \quad Q_{\mathbb{X}} = m[-1], \tag{0.39}$$

for $m \in \mathbb{Z}^+$. Combining this result with Freedman's classification, we can definitively state that any two smooth simply-connected four manifolds are homeomorphic if and only if their intersection forms have the same rank, signature, and parity.

While a proof of Donaldson's Theorem is beyond the scope of this thesis, it relies on a careful analysis of the instanton moduli space $\mathcal{M}_{k,g}$ of $\mathbb{X}$. Continuing his analysis of instantons, Donaldson also introduced his polynomial invariants of $\mathbb{X}$, which themselves, for $b_2^+ > 1$, are independent of the choice of metric. In other words, they are full diffeomorphism invariants and only depend on a choice of smooth structure. It is with such objects that one might hope to solve our still widely open third question of how to classify smooth structures.

The Donaldson polynomial invariants are themselves integrals of cohomology classes on the (compactified) instanton moduli space. These cohomology classes are parameterized through the *Donaldson map* which takes

$$\mu_D : H_n(\mathbb{X}, \mathbb{R}) \longrightarrow H^{4-n}(\mathcal{M}_{k,g}). \tag{0.40}$$

We then define the *Donaldson polynomial invariants* as

$$\mathfrak{P}_{\mathrm{D}}^{\ell,r}(p_0, \Sigma) = \int_{\mathcal{M}_{k,g}} \mu_D(p_0)^{\ell} \mu_D(\Sigma)^r, \tag{0.41}$$

for surface $\Sigma \in H_2(\mathbb{X}, \mathbb{R})$ and $p_0 \in \mathbb{X}$. We can also introduce formal variables p and s and write the *Donaldson generating function* as

$$\mathsf{Z}_{\mathrm{D}}[g, p, s] = \sum_{\ell, r \geq 0} \frac{p^{\ell}}{\ell!} \frac{s^r}{r!} \mathfrak{P}_{\mathrm{D}}^{\ell,r}, \tag{0.42}$$

As noted above, for $b_2^+ > 1$ the $\mathfrak{P}_{\mathrm{D}}^{\ell,r}$ are constant rational numbers which are independent of the choice of metric. Therefore, if two smooth structures on a smoothable four manifold give different values of $\mathfrak{P}_{\mathrm{D}}^{\ell,r}$, then it is clear that they have different smooth structures. Unfortunately, when $b_2^+ = 1$, they are only piecewise constant rational numbers on $\mathsf{Met}(\mathbb{X})$, and change value over walls of codimension one. To make matter worse, these invariants vanish completely on "half" of all smooth four manifolds. To see this, note that when $b_2^+ > 0$, the dimension of $\mathcal{M}_{k,g}$ in (0.24) for a generic metric is even only when $b_1 + b_2^+$ is also even. Since the integrand of the invariant is a $(4\ell + 2r)$-form on $\mathcal{M}_{k,g}$ it is clear that $\mathfrak{P}_{\mathrm{D}}^{\ell,r} = 0$ when $b_1 + b_2^+$ is odd i.e. for "half" of all smooth manifolds. We do not take this as a failure, but rather an opportunity to search for generalizations of these beautiful objects. But first, in order to practice good physical mathematics we need to understand how the current invariants are understood physically.

0.2 $\mathcal{N} = 2$, $d = 4$ Super Yang Mills

Ultimately, we will show that the Donaldson generating function is a correlation function for observables in the topological twisted $\mathcal{N} = 2$ super Yang-Mills theory. At present these words may have little meaning, so we begin with the simplest: “super.”

0.2.1 Supersymmetry

In the 1960s, there were a series of attempts to enlarge the number of symmetries in the Poincaré algebra in order to account for, at the time, unknown physical phenomena. Most of these attempts were put to rest by the famous *Coleman-Mandula Theorem* [9]. As a weighty no-go, it states that the possible symmetries of a well-behaved local relativistic quantum field theory are restricted to a direct product of the Poincaré group and a compact Lie group. Therefore the only allowed generators of symmetries are the energy-momentum operators P_μ, the Lorentz rotation generators $M_{\mu\nu}$, and some Lorentz invariant scalar charges in the Lie algebra of the compact Lie group, say B. At its core, the theorem prevents the non-trivial mixing of space-time and internal symmetries.

Fortunately for the world, or, at the very least, imaginary worlds, one can sidestep the restrictions of Coleman-Mandula by relaxing one condition. By allowing for symmetries that transform bosonic fields into fermionic ones and vice versa, one is permitted to include new symmetry generators that are in a spinor representation of the Poincaré group. Mathematically, this means we are considering Lie superalgebras as opposed to just a Lie algebra. These new spinorial objects are call *supercharges* and generate *supersymmetries.*[16]

In four dimensions, there are two inequivalent two dimensional spinor representations of the Poincaré group, which are called *Weyl spinors* in physics. Denoting these representations by their dimension, we write them as $\mathbf{2}$ and $\overline{\mathbf{2}}$. The generators are denoted by Q_A and $\overline{Q}_{\dot{A}}$, where A are the doublet indices of $\mathbf{2}$ and $\dot{A}$ are the doublet indices of $\overline{\mathbf{2}}$, which both run over the values 1 and 2. Hence, we have a total of four supercharges in the minimal theory, which we call $\mathcal{N}=1$ *supersymmetry.* They satisfy the anticommutation relations[17]

$$\{Q_A, \overline{Q}_{\dot{A}}\} = Q_A\overline{Q}_{\dot{A}} + \overline{Q}_{\dot{A}}Q_A = 2\sigma^{\mu}_{A\dot{A}}P_{\mu}, \tag{0.43}$$

$$\{Q_A, Q_B\} = \{\overline{Q}_{\dot{A}}, \overline{Q}_{\dot{B}}\} = 0, \tag{0.44}$$

where $\sigma^{\mu}_{A\dot{A}}$ is an intertwiner i.e. a homomorphism between represention, from the tensor product of our two inequivalent spinor representations of $\mathrm{Spin}(3,1) \cong \mathrm{SL}(2,\mathbb{C})$ to the vector representation of $\mathrm{SO}(3,1)$. In addition, the full $\mathcal{N}=1$ super Poincaré group has an extra internal $\mathrm{U}(1)_{\mathrm{R}}$ symmetry that only has non-trivial relations with the supercharges. Under it, the Q_As have charge 1 and the $Q_{\dot{A}}$s have charge -1. In general, internal symmetries which transform the supercharges are called *R-symmetries.*

We then write the even part of the full $\mathcal{N}=1$ super Poincaré algebra as

$$\mathsf{SP}^0_{\mathcal{N}=1} = \mathbb{R}^{3,1} \rtimes \mathfrak{so}(3,1) \oplus \mathfrak{u}(1)_{\mathrm{R}}. \tag{0.45}$$

The semi-direct product between translations and rotations reflects the non-trivial commutation relations between these symmetries.[18] Next, written in terms of rep-

[16]The canonical textbook for supersymmetry is [4], though a more modern perspective can be found in [21, 22].

[17]A cheap scrap of intuition is that supersymmetries are the "square root" of a translation.

[18]Walk forward and turn around and from the same initial position have your friend turn around

resentations of the Lorentz subalgebra of the Poincaré algebra and R-symmetry, the odd part is

$$\mathsf{SP}^1_{\mathcal{N}=1} = \mathbf{2}_1 \oplus \overline{\mathbf{2}}_{-1}. \tag{0.46}$$

Here, the subscript indicates the $\mathfrak{u}(1)_{\mathrm{R}}$ charge.

One can further extend this algebra to include an additional set of supercharges, which then leads to $\mathcal{N} = 2$ *supersymmetry*.[19] Introducing indices $i, j = 1, 2$ to distinguish between the two sets, our non-trivial supersymmetry relation is enhanced to

$$\{Q^i_A, \overline{Q}^j_{\dot{A}}\} = 2\varepsilon^{ij}\sigma^\mu_{\ A\dot{A}} P_\mu, \tag{0.47}$$

where ε^{ij} is the antisymmetric tensor, defined to have $\varepsilon^{12} = +1$. Further conventions and notations for spinors can be found in Appendix A.2. The even part of $\mathcal{N} = 2$ super Poincaré algebra is then

$$\mathsf{SP}^0_{\mathcal{N}=2} = \mathbb{R}^{3,1} \rtimes \mathfrak{so}(3,1) \oplus \mathfrak{su}(2)_{\mathrm{R}} \oplus \mathfrak{u}(1)_{\mathrm{R}}, \tag{0.48}$$

where we recognize a larger R-symmetry, which rotates the two sets of supercharges, thus giving i, j the meaning of indices of the fundamental represntion of $\mathfrak{su}(2)$. Including this representations of $\mathfrak{su}(2)_{\mathrm{R}}$, the odd part is now

$$\mathsf{SP}^1_{\mathcal{N}=2} = (\mathbf{2}; \mathbf{2})_1 \oplus (\overline{\mathbf{2}}; \mathbf{2})_{-1}. \tag{0.49}$$

This story has been up to now told in Minkowski space with a metric $\eta_{\mu\nu} = \mathrm{diag}[-1, 1, 1, 1]$. We are, of course, not interested in pseudo-Riemannian manifolds, so we must perform a Wick rotation to the desired Euclidean signature. The resulting

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and walk forward. You will find yourself at an advantage should you be about to duel.

[19] The reader is directed to the excellent resource [80].

$\mathcal{N}=2$ *super Euclidean algebra* then has even part

$$\mathsf{SE}^0_{\mathcal{N}=2} = \mathbb{R}^4 \rtimes (\mathfrak{su}(2)_+ \oplus \mathfrak{su}(2)_-) \oplus \mathfrak{su}(2)_{\mathrm{R}} \oplus \mathfrak{u}(1)_{\mathrm{R}} \tag{0.50}$$

where we have exploited the Lie algebra isomorphism $\mathfrak{so}(4) \cong \mathfrak{su}_+(2) \oplus \mathfrak{su}_-(2)$. The odd part is then

$$\mathsf{SE}^1_{\mathcal{N}=2} = (\mathbf{2}, \mathbf{1}; \mathbf{2})_1 \oplus (\mathbf{1}, \mathbf{2}; \mathbf{2})_{-1}, \tag{0.51}$$

where we understand the $(\mathbf{2}, \mathbf{1})$ representation of $\mathfrak{su}(2)_+ \oplus \mathfrak{su}(2)_-$ as the Wick rotated $\mathbf{2}$ representation of $\mathfrak{so}(3,1)$ and likewise $(\mathbf{1}, \mathbf{2})$ as $\overline{\mathbf{2}}$.

Having arrived at our final algebra, we now turn to one of its most important representations.

0.2.2 Vector Multiplet

Representations of supersymmetry can be understood as a collection of bosonic and fermionic fields, which we call a *multiplet*. Since we are interested in gauge theory, we need to understand the multiplet that contains a gauge connection $A \in \mathcal{A}(P)$ amongst its fields. This representation is known as the *vector multiplet.*

Other than the gauge connection, the fields in the multiplet are two complex scalar fields ϕ and $\overline{\phi}$, two Weyl fermions ψ and $\overline{\psi}$ called *gauginos*, and, in order to keep an equal number of bosonic and fermionic degrees of freedom, an auxiliary field D.[20] Each of these fields takes values in the adjoint representation of the gauge group G. Further, the Weyl fermions are in the $\mathbf{2}$ of $\mathfrak{su}(2)_{\mathrm{R}}$ while D is in the $\mathbf{3}$. Finally, each field has a $\mathfrak{u}(1)_{\mathrm{R}}$ charge. Collectively, this can all summarized in the table below.

[20]Counting, we actually find an extra bosonic degree of freedom which stems from the fact that ϕ and $\overline{\phi}$ are *not* complex conjugates in Euclidean space. The resolution of this issue comes down to a choice of "complex contour," though we do not dwell on this subject.

Field	Symbol	$\mathfrak{su}(2)_+ \oplus \mathfrak{su}(2)_- \oplus \mathfrak{su}(2)_{\mathrm{R}}$	$\mathfrak{u}(1)_{\mathrm{R}}$
Gauge Field	A	$(\mathbf{2},\mathbf{2};\mathbf{1})$	0
$\mathfrak{su}(2)_-$ Gaugino	$\overline{\psi}$	$(\mathbf{1},\mathbf{2};\mathbf{2})$	1
$\mathfrak{su}(2)_+$ Gaugino	ψ	$(\mathbf{2},\mathbf{1};\mathbf{2})$	-1
Complex Scalar	ϕ	$(\mathbf{1},\mathbf{1};\mathbf{1})$	2
$\overline{\text{Complex Scalar}}$	$\overline{\phi}$	$(\mathbf{1},\mathbf{1};\mathbf{1})$	-2
Auxiliary Field	D	$(\mathbf{1},\mathbf{1};\mathbf{3})$	0

Table 1: Vector multiplet fields.

We mention in passing that it is also possible to express this collection of fields in terms of two representations of the $\mathcal{N}=1$ algebra. Without diving into the business of superspace, we have the fields A and ψ contained in a $\mathcal{N}=1$ vector multiplet W_μ, and the fields $\overline{\psi}$, ϕ, and $\overline{\phi}$ contained in a so-called $\mathcal{N}=1$ chiral multiplet Φ. The auxiliary field D is shared between the two multiplet, and splits into two auxiliary $\mathfrak{su}(2)_{\mathrm{R}}$ doublets. We will very rarely appeal to this language, and refer the reader to [37, 56, 89] for a more thorough discussion of the superspace formalism.

In order to write the transformations of these fields under supersymmetric transformation, we introduce two anticommuting variational parameters $\boldsymbol{\epsilon}^{A}{}_{i}$ and $\overline{\boldsymbol{\epsilon}}^{\dot{A}}{}_{i}$. We then define the transformation of an arbitrary field $\mathcal{O}$ through

$$\delta\mathcal{O} = (\boldsymbol{\epsilon}^{A}{}_{i}Q^{i}_{A} + \overline{\boldsymbol{\epsilon}}^{\dot{A}}{}_{i}\overline{Q}^{i}_{\dot{A}})\mathcal{O} \tag{0.52}$$

Suppressing spinor indices for the moment, we have

$$\delta\phi = -\overline{\boldsymbol{\epsilon}}^{i}\overline{\psi}_i, \tag{0.53}$$

$$\delta\overline{\phi} = \boldsymbol{\epsilon}^{i}\psi_i, \tag{0.54}$$

$$\delta A_\mu = \boldsymbol{\epsilon}^{i}\sigma_\mu\overline{\psi}_i - \overline{\boldsymbol{\epsilon}}^{i}\sigma_\mu\psi_i, \tag{0.55}$$

$$\delta\psi_i = \overline{\boldsymbol{\epsilon}}_i\sigma^\mu D_\mu\overline{\phi} + \boldsymbol{\epsilon}_i[\phi,\overline{\phi}] + \boldsymbol{\epsilon}_i\sigma^{\mu\nu}F_{\mu\nu} - \boldsymbol{\epsilon}^{j}D_{ij}, \tag{0.56}$$

$$\delta\overline{\psi}_i = -\boldsymbol{\epsilon}_i\sigma^\mu D_\mu\phi - \overline{\boldsymbol{\epsilon}}_i[\phi,\overline{\phi}] + \overline{\boldsymbol{\epsilon}}_i\widetilde{\sigma}^{\mu\nu}F_{\mu\nu} + \overline{\boldsymbol{\epsilon}}^j D_{ij}, \tag{0.57}$$

$$\delta D_{ij} = 2\boldsymbol{\epsilon}_{(i}\sigma^\mu D_\mu\overline{\psi}_{j)} - 2\overline{\boldsymbol{\epsilon}}_{(i}\sigma^\mu D_\mu\psi_{j)} - \boldsymbol{\epsilon}_{(i}[\phi,\psi_{j)}] + \overline{\boldsymbol{\epsilon}}_{(i}[\overline{\phi},\overline{\psi}_{j)}]. \tag{0.58}$$

Here, we write D_μ for the gauge covariant derivative $D_A = d + A$ and $\sigma^{\mu\nu}$ or $\widetilde{\sigma}^{\mu\nu}$ for the projection onto the self-dual or anti-self-dual subspaces of two forms.

With just this data, our fields are only defined on $\mathbb{R}^4$. In order to extend their definition to an arbitrary four smooth manifold $\mathbb{X}$ with a principal bundle $P \longrightarrow \mathbb{X}$, we must provide the additional data of a principal bundle $P_{\mathrm{R}} \longrightarrow \mathbb{X}$ for the R-symmetry $\mathrm{SU}(2)_{\mathrm{R}}$. With the addition of this structure, we can now consider each field as a section of a particular bundle over $\mathbb{X}$. Case by case, A is a G-connection for P, ϕ and $\overline{\phi}$ are sections of $\mathsf{ad}\,P \otimes \mathbb{C}$, the adjoint bundle extended to the complex numbers , D is a section of $\mathsf{ad}\,P \otimes W$, the adjoint bundle tensored with a real rank three vector bundle associated to P_{R},, and ψ and $\overline{\psi}$ are sections of $\mathsf{ad}\,P \otimes S^\pm \otimes S_{\mathrm{R}}$, the adjoint bundle tensored with spin bundles $S^\pm$ on $\mathbb{X}$ and a spin bundle S_{R} associated to P_{R}.

The mathematically savvy will object to the last identification above, as the existence of spin bundles $S^\pm$ requires a *spin structure*. In fact, “most” four manifolds are not spin, and in such cases any attempt to lift transition function to a spin bundle will encounter a $\mathbb{Z}_2$ conflicts over intersections. The root of the issue is an obstruction to lifting a $\mathrm{SO}(4)$ bundle to its double cover $\mathrm{Spin}(4)$, which is realized in the *second Stiefel-Whitney class* $w_2(T\mathbb{X}) \in H^2(\mathbb{X},\mathbb{Z}_2)$ of the manifold. In a phrase, $w_2(\mathbb{X})$ measures the obstruction to trivializing $T\mathbb{X}$ over oriented surfaces embedded in $\mathbb{X}$. Thankfully, there is a nifty side-step to this obstruction for the case at hand. Should $w_2(T\mathbb{X}) \neq 0$, we take P_{R} to be a principal $\mathrm{SO}(3)$ bundle whose own second Stiefel-Whitney class satisfies $w_2(P_{\mathrm{R}}) = w_2(T\mathbb{X})$. Then, while on their own $S^\pm$ and S_{R} do not exist, the bundles $S^\pm \otimes S_{\mathrm{R}}$ do exist! Since the issue in the transition functions hinges on a $\mathbb{Z}_2$, the bundles can conspire to solve each other's problem; two wrongs do make a right in $\mathbb{Z}_2$. We will see how this work-around can be fully exploited when we conduct our twist.

0.2.3 $\mathcal{N} = 2$ Super Yang Mills Action

We now come to the end of our "untwisted" story with the $\mathcal{N} = 2$ super Yang-Mills action. It is defined as the "supersymmetric completion" of the Yang-Mills action (0.19), that is, the minimal extension of S_{YM} that it is invariant under the supersymmetric transformations of (0.53)-(0.58). Explicitly, we have

$$S^0_{\mathrm{SYM}} = \frac{1}{g_0^2}\int_{\mathbb{X}} d^4x\sqrt{g}\mathrm{Tr}\left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}D_{ij}D^{ij} - \mathcal{D}_\mu\overline{\phi}\mathcal{D}^\mu\phi + \overline{\psi}^i\sigma^\mu\mathcal{D}_\mu\psi_i \right.$$
$$\left. -\psi^i[\phi,\psi_i] - \overline{\psi}^i[\overline{\psi}_i,\overline{\phi}] - \frac{1}{2}[\phi,\overline{\phi}]^2\right], \quad (0.59)$$

where g_0 is the constant *bare gauge coupling.* Here the derivative $\mathcal{D}_\mu$ is not only gauge covariant, but is also metric covariant and therefore it contains a spin connection ω which splits into two components $\omega^+ = {\omega_\mu}^{AB}$ and $\omega^- = {\omega_\mu}^{\dot{A}\dot{B}}$, associated to its self-dual and anti-self-dual parts.[21] They can likewise be viewed as the spin connections for S^+ and S^- respectively. Further, the derivative is also covariant with respect to the R-symmetry transformations, and thus contains a connection for P_{R}, which we call ω_{R}.

In addition to (0.59), it is customary to include the topological term of the instanton number which we introduced in (0.18). The term is by itself invariant under our supersymmetry transformations, and we can freely add it to our action, leading

[21]We are being quite coy with indices here. For those who peruse Appendix A.2, we note that the spin connection ω has the index structure ${\omega_\mu}^{ab}$, where a and b are frame indices. We then have $\omega_\mu^{+AB} = {\sigma_{ab}}^{AB}{\omega_\mu}^{ab}$ and $\omega_\mu^{-\dot{A}\dot{B}} = {\widetilde{\sigma}_{ab}}^{\dot{A}\dot{B}}{\omega_\mu}^{ab}$. In curved space, our σ^μ should be written as ${e^\mu}_a\sigma^a$ to make this clear.

to

$$S_{\text{SYM}} = S^0_{\text{SYM}} + \frac{i\theta_0}{64\pi^2}\int_{\mathbb{X}} d^4x \text{Tr}\left[\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right], \tag{0.60}$$

where we have introduced the constant *theta angle* θ_0. We note in passing that the normalization here have been taken to align with [62, 71].

The theory just introduced is a specialization of the most general vector multiplet action. Written in terms of $\mathcal{N}=1$ superspace, the general action for an abelian gauge group takes the form

$$S_{\text{VM}} = \frac{1}{4\pi}\text{Im}\left[\int_{\widehat{\mathbb{X}}} d^4x\, d^4\theta \frac{\partial\mathcal{F}(\Phi)}{\partial\Phi}\overline{\Phi} + \frac{1}{2}\int_{\widehat{\mathbb{X}}} d^2\theta \frac{\partial^2\mathcal{F}}{\partial\Phi^2}W^\mu W_\mu\right], \tag{0.61}$$

where we write $\widehat{\mathbb{X}}$ for the $\mathcal{N}=2$ superspace of $\mathbb{X}$ which has coordinates $(x,\theta,\overline{\theta})$. The theory is entirely specified by the gauge invariant holomorphic function $\mathcal{F}(\Phi)$, which is called the *prepotential*. Further, it is constrained to be homogeneous of degree two, which means that $CF(\lambda\Phi) = \lambda^2\mathcal{F}(\Phi)$ for some $\lambda \in \mathbb{R}^+$. If we take

$$\mathcal{F}(\Phi) = \frac{\tau_0}{2}\Phi^2, \tag{0.62}$$

where τ_0 is the complex coupling constant

$$\tau_0 = \frac{4\pi i}{g_0^2} + \frac{\theta_0}{2\pi}. \tag{0.63}$$

Then S_{VM} with this prepotential will be identical to S_{SYM} with an abelian gauge group. Since this quadratic prepotential has the minimal amount of interaction between fields allowed by $\mathcal{N}=2$ supersymmetry, S_{SYM}, for any gauge group, is often called the *free theory*. We will also often refer to S_{SYM} as the *UV theory*.

The breakthrough of Seiberg and Witten in their work [76, 77] was to provide an exact low energy effective theory for the $G = \mathrm{SU}(2)$ $\mathcal{N} = 2$ super Yang Mills theory S_{SYM}. Flowing the UV theory into the low energy limit, the so-call *IR theory*, they discovered that the gauge symmetry is generically broken to U(1) at every point on the space of quantum vacua. Their solution provided a explicit computation of the prepotential $\mathcal{F}(\Phi)$ for a theory of the form (0.61). While we will not dive deeper into the fascinating world of Seiberg-Witten theory here, our generalization of Donaldson-Witten invariants will extend to the twisted IR theory, so it behooves us to be familiar with the form of its action.

0.3 The Twist

With $\mathcal{N} = 2$ supersymmetry addressed, we are now prepared to expose the "twist." Succinctly, twisting is the procedure of constraining a thoery to a particular gravitational background whereon the correlation functions are independent of the metric, i.e. topological. In addition, the resulting supersymmetry algebra contains a scalar supercharge $\mathcal{Q}$ which satisfies $\mathcal{Q}^2 = 0$ on gauge invariant objects, therefore allowing for one to speak of the cohomology of the theory. We will provide a few perspectives on this process.

For a first taste, recall from our discussion of the vector multiplet that our ability to put the $\mathcal{N} = 2$ super Yang-Mills theory on an arbitrary four manifold $\mathbb{X}$ is predicated by a *choice* of $\mathrm{SU}(2)_R$ principal bundle P_{R} and a *choice* of connection ω_{R}. As Witten realized, this choice can make a world of difference. Suppose we take P_{R} to be isomorphic to the SO(3) bundle $P^+ \to \mathbb{X}$ associated with self-dual forms on $\mathbb{X}$, and then further make the choice that the connections on each bundle are isomorphism, i.e. $\omega^+ \cong \omega_{\mathrm{R}}$. Then, looking at S_{SYM}, we realize all $\omega^\pm$ and ω_{R} dependence is contained in the $\overline{\psi}^i\sigma^\mu D_\mu\psi_i$ term. Expanding the derivative, we have

$$\mathcal{D}_\mu\psi^{iA} = \partial_\mu\psi^{iA} + [A_\mu, \psi^{iA}] + \frac{1}{2}\omega_\mu^{+AB}\psi^i{}_B - \frac{1}{2}\omega_{\mathrm{R}\mu}{}^{ij}\psi_j{}^A, \tag{0.64}$$

Exploiting the isomorphism $P_{\mathrm{R}} \cong P^+$ to its fullest extend, we have

$$\omega_{\mathrm{R}\mu}{}^{ij}\delta_j^A = \omega_\mu^{+AB}\delta_B^i \tag{0.65}$$

where we are identifying the $\mathfrak{su}(2)_{\mathrm{R}}$ and $\mathfrak{su}(2)_+$ indices. With this identification the last two terms of (0.64) above cancel, quite remarkably eliminating all dependence on $\omega^\pm$ and ω_{R} in S_{SYM}. This isomorphism $\omega^+ \cong \omega_{\mathrm{R}}$ should be understood as changing the theory's coupling to gravity, since ω^+ is indeed dependent on the metric on $\mathbb{X}$.

For a more systematic and global perspective, consider two subgroups of the super Euclidean group given by

$$G_0 = (\mathrm{SU}(2)_+ \times \mathrm{SU}(2)_- \times \mathrm{SU}(2)_{\mathrm{R}})/\mathbb{Z}_2, \qquad \text{and} \qquad G_1 = (\mathrm{SU}(2)_+ \times \mathrm{SU}(2)_-)/\mathbb{Z}_2, \tag{0.66}$$

where both quotients are by the central subgroup which acts as $(-\mathbb{1}, -\mathbb{1}, -\mathbb{1})$ and $(-\mathbb{1}, -\mathbb{1})$ respectively. Ignoring the abelian R-symmetry, G_0 is the group under which our supercharges transform, with the spinor representation of

$$(\mathbf{2}, \mathbf{1}; \mathbf{2}) \oplus (\mathbf{1}, \mathbf{2}; \mathbf{2}). \tag{0.67}$$

The subgroup G_1 is the structure group of the tangent bundle $T\mathbb{X}$, that is, the rotation group, and itself dictates the behavior of the theory's gravitational coupling. We now introduce the injective homomorphism $G_1 \hookrightarrow G_0$ defined by

$$[(g_1, g_2)] \longrightarrow [(g_1, g_2, g_1)], \tag{0.68}$$

where we are working with equivalence classes under the $\mathbb{Z}_2$ quotient. Now, given a Riemannian metric, we can specialize to G_0-bundles with R-symmetry and Levi-Civita connections which pullback to G_1-bundles with a single Levi-Civita connection under the homeomorphism. This is likewise equivalent to defining a new $\mathrm{SU}(2)'_+$ as the diagonal subgroup of $\mathrm{SU}(2)_+ \times \mathrm{SU}(2)_\mathrm{R}$, and then taking $(\mathrm{SU}(2)'_+ \times \mathrm{SU}(2)_-)/\mathbb{Z}_2$ as the new structure group for $T\mathbb{X}$. In mathematical parlance, we are conducting a *reductive of the structure group* associated to the homomorphism (0.68). In this approach, the rationale for identifying the $\mathfrak{su}(2)_\mathrm{R}$ and $\mathfrak{su}(2)_+$ indices in (0.65) is manifest.

We can pullback our supercharges under this map to obtain a new representation

$$(\mathbf{1}, \mathbf{1}) \oplus (\mathbf{3}, \mathbf{1}) \oplus (\mathbf{2}, \mathbf{2}) \tag{0.69}$$

We thus see that our original eight supercharges have been "twisted" into a scalar supercharge $\mathcal{Q}$, a vector supercharge $\mathsf{K}_{A\dot{A}}$, and a self-dual supercharge $\mathcal{Q}^+_{AB}$. Note that none of these are spinor representations, and therefore the resulting theory has no need for a spin structure and can be defined on any smooth four manifold $\mathbb{X}$. Unfortunately, in general, a smooth four manifold does not admit non-vanishing vectors or self-dual forms, so $\mathsf{K}_{A\dot{A}}$ and $\mathcal{Q}^+_{AB}$ will not exist. Thankfully, all smooth four manifolds allow for a non-vanishing scalar field $\mathcal{Q}$.

While escaping the need for extra structure on $\mathbb{X}$ is a great boon, we have yet to see the real power of the twist. To do so, we turn to the twisted vector multiplet and its transformations under our new scalar supercharge $\mathcal{Q}$.

0.3.1 Twisted Vector Multiplet

Let's take a look at the resulting twisted vector multiplet representation. In the trenches, the twist follows a very simple program. First, utilizing the index indentification perspective of (0.65), every time we find a $\mathfrak{su}(2)_{\mathrm{R}}$, index i, j we replace it with a new $\mathfrak{su}(2)_{+}$ index A, B. Next, since the twist theory is devoid of any spinors, we will exercise to the fullest extend our intertwiners σ_μ and $\sigma_{\mu\nu}$, arriving at objects with exclusively spatial indices μ, ν, &c.

We begin by noting that any fields that transformed trivially under the $\mathrm{SU}(2)_{\mathrm{R}}$ symmetry, namely the scalar fields ϕ and $\overline{\phi}$ and the gauge connection A_μ, do not change. Nevertheless, to align with the literature, we will write $\overline{\phi}$ as λ.[22] Meanwhile the Weyl fermions experience radical developments. First, $\overline{\psi}_{\dot{A}i}$ now becomes $\overline{\psi}_{\dot{A}B}$, so we define[23]

$$\psi_\mu = \sigma_\mu^{A\dot{A}} \overline{\psi}_{A\dot{A}}. \tag{0.70}$$

Turning to ψ_{Ai}, we arrive at ψ_{AB} which splits into symmetric and antisymmetric parts as

$$\psi_{AB} = \psi_{(AB)} + \frac{1}{2}\psi\epsilon_{AB}, \tag{0.71}$$

where $\psi_{(AB)}$ is the symmetric part and ψ is the trace. We then define

$$\chi_{\mu\nu} = \sigma_{\mu\nu}^{AB}\psi_{(AB)}, \qquad \text{and} \qquad \eta = \frac{1}{2}\psi, \tag{0.72}$$

[22] It is worth mentioning that in the physical theory, prior to the Wick rotation, the fields ϕ and $\overline{\phi}$ are complex conjugates of each other. In the present theory, we do not recognize this relation and consider ϕ and λ to be independent fields, thus the change in notation. If we were to require them to be complex conjugates of each other, then the final objects in our analysis, the Donaldson-Witten invariants would not be real.

[23] We lament the notation, but the literature is rigid in its preferences. It is crucially important not to confuse this with $\overline{\psi}_{Ai}$.

giving us a self-dual fermionic two form $\chi_{\mu\nu}$ and a fermionic scalar η. Finally, the auxiliary field D_{ij} originally transformed in the $\mathbf{3}$ of $\mathfrak{su}(2)_{\mathrm{R}}$, so as D_{AB}, we can express it as a self-dual two-form and write

$$D_{\mu\nu} = \sigma^{AB}_{\mu\nu} D_{AB}. \tag{0.73}$$

At times it is preferable to use an alternative auxiliary field definition of

$$H_{\mu\nu} = F^+_{\mu\nu} - D_{\mu\nu}, \tag{0.74}$$

which we note is still very much a self-dual field. The twisted vector multiplet field content is summarized in the table below.

Field	Symbol	$\mathfrak{su}(2)'_+ \oplus \mathfrak{su}(2)_-$	Bundle	$\mathfrak{u}(1)_{\mathrm{R}}$
Gauge Field	A_μ	$(\mathbf{2},\mathbf{2})$	$\mathcal{A}(P)$	0
Vector Gaugino	ψ_μ	$(\mathbf{2},\mathbf{2})$	$\Pi\Omega^1(\mathbb{X}, \mathsf{ad}\, P)$	1
Scalar Field	ϕ	$(\mathbf{1},\mathbf{1})$	$\Omega^0(\mathbb{X}, \mathsf{ad}\, P)$	2
Scalar Field	λ	$(\mathbf{1},\mathbf{1})$	$\Omega^0(\mathbb{X}, \mathsf{ad}\, P)$	-2
Scalar Gaugino	η	$(\mathbf{1},\mathbf{1})$	$\Pi\Omega^0(\mathbb{X}, \mathsf{ad}\, P)$	-1
Self-Dual Gaugino	$\chi_{\mu\nu}$	$(\mathbf{3},\mathbf{1})$	$\Pi\Omega^{2,+}_g(\mathbb{X}, \mathsf{ad}\, P)$	-1
Self-Dual Auxiliary Field	$H_{\mu\nu}/D_{\mu\nu}$	$(\mathbf{3},\mathbf{1})$	$\Omega^{2,+}_g(\mathbb{X}, \mathsf{ad}\, P)$	0

Table 2: Twisted vector multiplet.

Above, we denote the *superspace* of a bundle $E \longrightarrow \mathbb{X}$ as ΠE, which indicates that the fibres are considered odd i.e. fermionic. We also would be remiss not to mention that both $\Pi\Omega^{2,+}_g(\mathbb{X}, \mathsf{ad}\, P)$ and $\Omega^{2,+}_g(\mathbb{X}, \mathsf{ad}\, P)$ depend on a choice of metric $g \in \mathsf{Met}\mathbb{X}$ as can be seen in the definition (0.10). This will be a source of momentarily turmoil when we move to our generalization. Finally, since everything is valued in $\mathsf{ad}\, P$, we will often omit it.

Next, turning to the twisted transformation laws, our original (0.53)-(0.58), restricted to just the scalar supercharge $\mathcal{Q}$, gives us[24]

$$\delta\big|_{\mathcal{Q}}\phi = 0, \tag{0.75}$$

$$\delta\big|_{\mathcal{Q}}\lambda = \boldsymbol{\epsilon}\eta, \tag{0.76}$$

$$\delta\big|_{\mathcal{Q}}A_\mu = \boldsymbol{\epsilon}\psi_\mu, \tag{0.77}$$

$$\delta\big|_{\mathcal{Q}}\eta = \boldsymbol{\epsilon}[\phi, \lambda], \tag{0.78}$$

$$\delta\big|_{\mathcal{Q}}\chi_{\mu\nu} = \boldsymbol{\epsilon}F^+_{\mu\nu} - \boldsymbol{\epsilon}D_{\mu\nu} \qquad \text{or} \qquad \delta\big|_{\mathcal{Q}}\chi_{\mu\nu} = \boldsymbol{\epsilon}H_{\mu\nu}, \tag{0.79}$$

$$\delta\big|_{\mathcal{Q}}\psi_\mu = -\boldsymbol{\epsilon}D_\mu\phi, \tag{0.80}$$

$$\delta\big|_{\mathcal{Q}}D_{\mu\nu} = 2\boldsymbol{\epsilon}(D_{[\mu}\psi_{\nu]})^+ - \boldsymbol{\epsilon}[\phi, \chi_{\mu\nu}], \qquad \text{or} \qquad \delta\big|_{\mathcal{Q}}H_{\mu\nu} = \boldsymbol{\epsilon}[\phi, \chi_{\mu\nu}]. \tag{0.81}$$

Reversing our definition for the transformation (0.52) and extracting the parameter $\boldsymbol{\epsilon}$, we can rewrite the above as

$$\mathcal{Q}\phi = 0, \tag{0.82}$$

$$\mathcal{Q}\lambda = \eta, \tag{0.83}$$

$$\mathcal{Q}A_\mu = \psi_\mu, \tag{0.84}$$

$$\mathcal{Q}\eta = [\phi, \lambda], \tag{0.85}$$

$$\mathcal{Q}\chi_{\mu\nu} = F^+_{\mu\nu} - D_{\mu\nu}, \qquad \text{or} \qquad \mathcal{Q}\chi_{\mu\nu} = H_{\mu\nu} \tag{0.86}$$

$$\mathcal{Q}\psi_\mu = -D_\mu\phi, \tag{0.87}$$

$$\mathcal{Q}D_{\mu\nu} = 2(D_{[\mu}\psi_{\nu]})^+ - [\phi, \chi_{\mu\nu}], \qquad \text{or} \qquad \mathcal{Q}H_{\mu\nu} = [\phi, \chi_{\mu\nu}]. \tag{0.88}$$

[24] Here we are working with fields rescaled relative to the original work of Witten. Denoting the fields of [85] with a superscript W, we have $\psi \mapsto i\psi^W$, $\phi \mapsto i\phi^W$, $\lambda \mapsto -\frac{i}{4}\lambda^W$, $\eta \mapsto \frac{1}{2}\eta$, and $\chi_{\mu\nu} \mapsto \frac{1}{2}\chi^W_{\mu\nu}$. Since the theory there is on-shell, one must also note a use of the equation of motion $D_{\mu\nu} = 0$.

With this approach, we see that

$$\mathcal{Q}^2 = \delta_\phi, \tag{0.89}$$

where δ_ϕ is a left-action gauge transformation by ϕ. In a physical theory, we exclusively work with gauge invariant objects, and therefore our scalar supercharge $\mathcal{Q}$ in this context is a nilpotent differential. Indeed, as we will discuss in the main body of our work, $\mathcal{Q}$ is the differential for the quotient space $\mathcal{A}/\mathcal{G}$ and allows us to consider the equivariant cohomology classes of $H_{\mathcal{G}}(\mathcal{A}(P))$, but all in due time. At present, it's time to twist the action.

0.3.2 Twisted Super Yang-Mills Action

Twisting the fields of our $\mathcal{N} = 2$ super Yang-Mills action (0.60), we arrive at our *twisted UV* $\mathcal{N} = 2$ *super Yang-Mills action*

$$\begin{aligned} S_{\mathrm{UV}} = \frac{1}{g_0^2} \int_{\mathbb{X}} d^4x \sqrt{g} \mathrm{Tr} \Big[&\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} D_{\mu\nu} D^{\mu\nu} + 2(D_\mu \psi_\nu)\chi^{\mu\nu} + \frac{1}{2}\chi_{\mu\nu}[\phi, \chi^{\mu\nu}] \\ &- 2\eta D_\mu \psi^\mu - 2\lambda[\psi_\mu, \psi^\mu] + 2\lambda D_\mu D^\mu \phi - 2\phi[\eta, \eta] - 2[\phi, \lambda]^2 \Big] \\ + \frac{i\theta_0}{64\pi^2} \int_{\mathbb{X}} d^4x \mathrm{Tr} \left[\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \right]. \end{aligned} \tag{0.90}$$

Here, since the action is entirely quadratic in the auxiliary field $D_{\mu\nu}$, its equation of motion is simply $D_{\mu\nu} = 0$. In our conventions, that

$$\frac{1}{4}\sqrt{g} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2}\sqrt{g} F^+_{\mu\nu} F^{\mu\nu}_+ - \frac{1}{8}\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}, \tag{0.91}$$

so we can rewrite our action as

$$S_{\mathrm{UV}} = \frac{1}{g_0^2} \int_{\mathbb{X}} d^4x \sqrt{g} \mathrm{Tr} \Big[\frac{1}{2} F^+_{\mu\nu} F^{\mu\nu}_+ - \frac{1}{2} D_{\mu\nu} D^{\mu\nu} + 2(D_\mu \psi_\nu)\chi^{\mu\nu} + \frac{1}{2}\chi_{\mu\nu}[\phi, \chi^{\mu\nu}]$$

$$-2\eta D_\mu\psi^\mu - 2\lambda[\psi_\mu,\psi^\mu] + 2\lambda D_\mu D^\mu\phi - 2\phi[\eta,\eta] - 2[\phi,\lambda]^2\Big]$$
$$+\frac{i\tau_0}{32\pi}\int_{\mathbb{X}} d^4x \mathrm{Tr}\left[\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right]. \tag{0.92}$$

where we recall the definition of our complex coupling τ_0 in (0.63). Further, and quite crucially, we can now write the entire twisted action as[25]

$$S_{\mathrm{UV}} = \mathcal{Q}V_{\mathrm{UV}} + \frac{i\tau_0}{8\pi}\int_{\mathbb{X}} \mathrm{Tr}F_A\wedge F_A, \tag{0.93}$$

where

$$V_{\mathrm{UV}} = \frac{1}{g_0^2}\int_{\mathbb{X}} d^4x\sqrt{g}\mathrm{Tr}\left[\frac{1}{2}(F^+_{\mu\nu}+D_{\mu\nu})\chi^{\mu\nu} - 2\lambda D_\mu\psi^\mu - 2\eta[\phi,\lambda]\right]. \tag{0.94}$$

In this formulation, it is manifest that

$$\mathcal{Q}S_{\mathrm{UV}} = 0, \tag{0.95}$$

since V_{UV} is a gauge invariant object and $\mathcal{Q}^2=\delta_\phi$. The topological itself closes under $\mathcal{Q}$, since it results in a total derivative.

While the fact that our action can be written as a $\mathcal{Q}$-exact part and a non-exact topological term is remarkable on its own, the true beauty of the theory is hidden in the metric dependence. We have already seen that our choice of isomorphism between the principal bundles P_{R} and P^+ leads all dependence on the spin connection ω to drop out of the action, but there is still explicit metric dependence in the action. To see it, we take an infinitesimal change $g^{\mu\nu}\longrightarrow g^{\mu\nu}+\delta g^{\mu\nu}$ and compute the energy-

[25] Our convetions for differential forms can be found in Appendix A.1.

momentum tensor of the theory through

$$\delta_g S_{\mathrm{UV}} = \frac{1}{2}\int_{\mathbb{X}} d^4x\sqrt{g}\delta g^{\mu\nu}T^{\mathrm{UV}}_{\mu\nu}. \tag{0.96}$$

We then find

$$\begin{aligned} T^{\mathrm{UV}}_{\mu\nu} = \mathrm{Tr}\Big[&\frac{1}{2}F_{\mu\rho}F_{\nu}{}^{\rho} + \frac{1}{2}F_{\nu\rho}F_{\nu}{}^{\rho} - \frac{1}{4}g_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma} \\ &+(D_{[\mu}\psi_{\rho]})\chi_{\nu}{}^{\rho} + (D_{[\nu}\psi_{\rho]})\chi_{\mu}{}^{\rho} - \frac{1}{2}g_{\mu\nu}(D_\rho\psi_\sigma)\chi^{\rho\sigma} \\ &+2(D_\mu\eta)\psi_\nu + 2(D_\nu\eta)\psi_\mu - 2g_{\mu\nu}(D_\sigma\eta)\psi^\sigma \\ &-2(D_\mu\lambda)(D_\nu\phi) + 2(D_\nu\lambda)(D_\mu\phi) + 2g_{\mu\nu}(D_\rho\lambda)(D^\rho\phi) \\ &-2\lambda[\psi_\mu,\psi_\nu] + 2g_{\mu\nu}\lambda[\psi_\rho,\psi^\rho] + 2g_{\mu\nu}\phi[\eta,\eta] + 2g_{\mu\nu}[\phi,\lambda]^2\Big]. \end{aligned} \tag{0.97}$$

In this computation we have made use of the fact that $\mathbb{X}$ is closed, allowing us to integrate by parts and avoid any variations of our Levi-Civita, or metric, connection. We have also made use of the identity

$$\delta_g\sqrt{g} = -\frac{1}{2}\sqrt{g}g_{\mu\nu}\delta g^{\mu\nu}. \tag{0.98}$$

Finally, the self-dual fields have implicit metric dependence which must be preserved under a change in the metric. For example, we have

$$\delta_g D_{\mu\nu} = -\frac{1}{4}g_{\rho\sigma}\delta g^{\rho\sigma}D_{\mu\nu} + \frac{1}{2}\sqrt{g}\epsilon_{\mu\nu\rho\sigma}\delta g^{\rho\rho'}g^{\sigma\sigma'}D_{\rho'\sigma'}. \tag{0.99}$$

Variations of this sort will soon become commonplace and they are thoroughly explored in Appendix D along with a myriad of other useful identities. Inspecting (0.97), we see that it can be written

$$T^{\rm UV}_{\mu\nu} = \mathcal{Q}\Lambda^{\rm UV}_{\mu\nu}, \tag{0.100}$$

where

$$\Lambda^{\rm UV}_{\mu\nu} = \mathrm{Tr}\Big[\frac{1}{2}F_\mu{}^\rho\chi_{\nu\rho} + \frac{1}{2}F_\nu{}^\rho\chi_{\mu\rho} - \frac{1}{4}g_{\mu\nu}F^{\rho\sigma}\chi_{\rho\sigma} \\ +2(D_\mu\lambda)\psi_\nu + 2(D_\nu\lambda)\psi_\nu - 2g_{\mu\nu}(D_\sigma\lambda)\psi^\sigma + 2g_{\mu\nu}\eta[\phi,\lambda]\Big]. \tag{0.101}$$

In the next section we will see how the fact that the energy-momentum tensor is $\mathcal{Q}$-exact means that the theory is truly independent of the choice metric and that all correlations of its observables are diffeomorphism invariants of $\mathbb{X}$.

Before we turn to the exciting conclusion to this tale, we quickly display the twisted IR theory. Here, we want to write the explicit twisted action (0.61) for a U(1) vector mulitplet with an arbitrary prepotential $\mathcal{F}$. With an abelian gauge group, our transformations (0.82)-(0.88) now take the form[26]

$$\mathcal{Q}A_\mu = \psi_\mu, \qquad \mathcal{Q}\psi_\mu = -\nabla_\mu a, \tag{0.102}$$

$$\mathcal{Q}a = 0, \tag{0.103}$$

$$\mathcal{Q}\overline{a} = \eta, \qquad \mathcal{Q}\eta = 0, \tag{0.104}$$

$$\mathcal{Q}\chi_{\mu\nu} = F^+_{\mu\nu} - D_{\mu\nu}, \qquad \mathcal{Q}D_{\mu\nu} = 2(\nabla_{[\mu}\psi_{\nu]})^+, \tag{0.105}$$

where for historic reasons we write a for ϕ and $\overline{a}$ for λ. Further, we define the arbitrary complex couplings

$$\tau = \frac{\partial^2\mathcal{F}(a)}{\partial a^2}, \qquad \text{and} \qquad \overline{\tau} = \frac{\partial^2\overline{\mathcal{F}}(\overline{a})}{\partial \overline{a}^2}. \tag{0.106}$$

[26]To obtain the transformations and action of [71] (and the more general case of [62]) exactly, denoting the fields therein with superscript u, there is a rescaling of $a \mapsto 4\sqrt{2}a^u$, $\overline{a} \mapsto \frac{1}{2\sqrt{2}}\overline{a}^u$, $\eta \mapsto 2i\eta^u$ and $\chi_{\mu\nu} \mapsto -i\chi^u_{\mu\nu}$.

We, again for those who read footnotes, recognize that in the Euclidean theory, twisted or not, $\mathcal{F}$ and $\overline{\mathcal{F}}$ are independent functions and need not be complex conjugates. For this reason, we take it as a definition that

$$\mathrm{Im}\,\tau = \frac{\tau - \overline{\tau}}{2i}. \tag{0.107}$$

Then, our *twisted IR* $\mathcal{N} = 2$ *super Yang-Mills action* is

$$\begin{aligned} S_{\mathrm{IR}} = \frac{i}{2^4\pi}\int_{\mathbb{X}} d^4x\sqrt{g}\Bigg[&\frac{1}{2}\overline{\tau}F^+_{\mu\nu}F^{\mu\nu}_+ - \frac{1}{2}\tau F^-_{\mu\nu}F^{\mu\nu}_- + 4i\mathrm{Im}\tau(\nabla_\sigma a)(\nabla^\sigma \overline{a}) + i\mathrm{Im}\tau D_{\mu\nu}D^{\mu\nu} \\ &+2\tau\psi_\sigma\nabla^\sigma\eta - 2\overline{\tau}\eta\nabla_\sigma\psi^\sigma - 2\tau\psi_\mu\nabla_\nu\chi^{\mu\nu} + 2\overline{\tau}(\nabla_\mu\psi_\nu)^+\chi^{\mu\nu} \\ &+\frac{1}{2}\frac{\partial\overline{\tau}}{\partial\overline{a}}\eta(F^+_{\mu\nu} + D_{\mu\nu})\chi^{\mu\nu} - \frac{\partial\tau}{\partial a}\psi_\mu\psi_\nu(F^{\mu\nu}_- + D^{\mu\nu}) \\ &+\frac{1}{12}\sqrt{g}^{-1}\frac{\partial^2\tau}{\partial a^2}\epsilon^{\mu\nu\rho\sigma}\psi_\mu\psi_\nu\psi_\rho\psi_\sigma + \mathcal{Q}\left(\frac{i}{12}\frac{\partial\overline{\tau}}{\partial\overline{a}}\chi_\mu{}^\rho\chi^{\mu\sigma}\chi_{\rho\sigma}\right)\Bigg]. \end{aligned} \tag{0.108}$$

Similar to the UV theory, this action splits into an $\mathcal{Q}$-exact part and a non-exact topological term. We write

$$S_{\mathrm{IR}} = \mathcal{Q}(V_{\mathrm{IR}} + \overline{V}_{\mathrm{IR}}) + \mathsf{C}_{\mathrm{IR}}, \tag{0.109}$$

where

$$V_{\mathrm{IR}} = \frac{i}{2^4\pi}\int_{\mathbb{X}} d^4x\sqrt{g}\left[-\frac{1}{2}\tau(F^+_{\mu\nu} + D_{\mu\nu})\chi^{\mu\nu} - 2\tau\psi_\sigma\nabla^\sigma\overline{a} + \frac{\partial\tau}{\partial a}\psi_\mu\psi_\nu\chi^{\mu\nu}\right], \tag{0.110}$$

$$\overline{V}_{\mathrm{IR}} = \frac{i}{2^4\pi}\int_{\mathbb{X}} d^4x\sqrt{g}\left[\frac{1}{2}\overline{\tau}(F^+_{\mu\nu} + D_{\mu\nu})\chi^{\mu\nu} - 2\frac{\partial\overline{\mathcal{F}}}{\partial\overline{a}}\nabla_\sigma\psi^\sigma + \frac{i}{12}\frac{\partial\overline{\tau}}{\partial\overline{a}}\chi_\mu{}^\rho\chi^{\mu\sigma}\chi_{\rho\sigma}\right], \tag{0.111}$$

and

$$\mathsf{C}_{\rm IR} = \frac{i}{2^4\pi}\int_{\mathbb{X}} d^4x \left[\frac{\tau}{4}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} - \frac{1}{2}\frac{\partial\tau}{\partial a}\epsilon^{\mu\nu\rho\sigma}\psi_\mu\psi_\nu F_{\rho\sigma} + \frac{1}{12}\frac{\partial^2\tau}{\partial a^2}\epsilon^{\mu\nu\rho\sigma}\psi_\mu\psi_\nu\psi_\rho\psi_\sigma\right]. \tag{0.112}$$

Here, the action is, of course, $\mathcal{Q}$-closed as

$$\mathcal{Q}S_{\rm IR} = 0, \tag{0.113}$$

though the closure of the topological term is a pinch more involved, as

$$\mathcal{Q}\mathsf{C}_{\rm IR} = \frac{i}{2^4\pi}\int_{\mathbb{X}} d^4x\nabla_\mu\left[\tau\epsilon^{\mu\nu\rho\sigma}\psi_\nu F_{\rho\sigma} - \frac{1}{3}\frac{\partial\tau}{\partial a}\epsilon^{\mu\nu\rho\sigma}\psi_\nu\psi_\rho\psi_\sigma\right] = 0, \tag{0.114}$$

since $\mathbb{X}$ is closed.

We can also observe that when we specialize to an abelian gauge group and take the quadratic prepotentials

$$\mathcal{F}(a) = \frac{\tau_0}{2}a^2, \qquad \text{and} \qquad \overline{\mathcal{F}}(\overline{a}) = \frac{\overline{\tau}_0}{2}\overline{a}^2, \tag{0.115}$$

where we do actually mean the complex conjugate of the complex coupling constant, then the action $S_{\rm IR}$ agrees with $S_{\rm UV}$.

0.3.3 Donaldson-Witten Invariants

We are now at long last ready to construct the Donaldson-Witten invariants. To start, we will investigate the properties of the simplest of the invariants, the so-called *partition function*. It is defined as

$$\mathsf{Z}_{\rm W}[g] = \int [d\mathrm{VM}]e^{-S_{\rm UV}}, \tag{0.116}$$

where we have introduced the vector mutliplet path integral measure[27]

$$[d\mathrm{VM}] = [dA\, d\phi\, d\lambda\, dD\, d\psi\, d\eta\, d\chi]. \tag{0.117}$$

In the most important computation of the twisted theory, under a change in the metric, we find

$$\delta_g \mathsf{Z}_{\mathrm{W}}[g] = \delta_g \left[\int [d\mathrm{VM}] e^{-S_{\mathrm{UV}}} \right] \tag{0.118}$$

$$= \int [d\mathrm{VM}] \delta_g \left(e^{-S_{\mathrm{UV}}} \right) \tag{0.119}$$

$$= \int [d\mathrm{VM}] \left(\frac{1}{2} \int_{\mathbb{X}} d^4x \sqrt{g(x)} \delta g^{\mu\nu}(x) T^{\mathrm{UV}}_{\mu\nu}(x) \right) e^{-S_{\mathrm{UV}}} \tag{0.120}$$

$$= \int [d\mathrm{VM}] \mathcal{Q} \left[\frac{1}{2} \int_{\mathbb{X}} d^4x \sqrt{g(x)} \delta g^{\mu\nu}(x) \Lambda^{\mathrm{UV}}_{\mu\nu}(x) \right] e^{-S_{\mathrm{UV}}} \tag{0.121}$$

$$= \int [d\mathrm{VM}] \mathcal{Q} \left[\frac{1}{2} \int_{\mathbb{X}} d^4x \sqrt{g(x)} \delta g^{\mu\nu}(x) \Lambda^{\mathrm{UV}}_{\mu\nu}(x) e^{-S_{\mathrm{UV}}} \right] \tag{0.122}$$

$$= 0. \tag{0.123}$$

We have been rather pendantic as nearly every lines holds an important lesson. In the first line, we rely on the fact that there are no diffeomorphism anamolies in four dimensions [1]. If there were, we would have not been able to move our metric variation through the vector multiplet measure. The next line is definitional, but allows us to exploit the hallmark relation of (0.100) in moving from (0.120) to (0.121). As mentioned prior, the exactness of the energy-momentum tensor is the crux of the twist. Next, since $\mathcal{Q}S_{\mathrm{UV}} = 0$ on account of the residual supersymmetry, we have the next line from (0.121) to (0.122). Finally, the final line is a lie, that is to say

[27] We will not concern ourselves here with the "mythic" properties of this measure. Suffice to say that a rigorous definition of the path integral has long eluded mathematicians. It is the author's perspective that any physical mathematician that accepts the path integral into their hearts will be met with a fount of results. As is often said, "Too much rigor leads to rigor mortis."

it is only formally true. It requires an integration by parts over the entirety of the vector multiplet field space, which is known to fail for $\mathbb{X}$ with $b_2^+ = 1$, where there are contributions from boundary terms. In this case, $\mathsf{Z}_{\mathrm{DW}}[g]$ is only piece-wise constant on $\mathsf{Met}(\mathbb{X})$. Though, at present unknown, there is also an expectation of continuous metric dependence for $b_2^+ = 0$ [75] . Still, for $b_2^+ > 1$, the final identity holds and the partition function is entirely independent of the choice of metric!

We can also consider correlation functions of gauge invariant *observables* of our theory. For such an object $\mathcal{O}$, we define its expectation value

$$\langle \mathcal{O} \rangle_{\mathrm{UV}} = \int [d\mathrm{VM}] \mathcal{O}\, e^{-S_{\mathrm{UV}}}. \tag{0.124}$$

Note that, expect in cases where the field space integration by parts fails, if $\mathcal{O}$ is $\mathcal{Q}$-exact, say $\mathcal{O} = \mathcal{Q}\mathcal{V}$, we have

$$\langle \mathcal{O} \rangle_{\mathrm{UV}} = \langle \mathcal{Q}\mathcal{V} \rangle_{\mathrm{UV}} = 0. \tag{0.125}$$

That is, $\mathcal{Q}$-exact operators decouple from the theory and we can form equivalence classes under which two observables are equal if they only differ by the addition of a $\mathcal{Q}$-exact term. In addition, if an observable has no metric dependence and is $\mathcal{Q}$-closed, we have

$$\delta_g \langle \mathcal{O} \rangle_{\mathrm{UV}} = \frac{1}{2} \int_{\mathbb{X}} d^4x \sqrt{g(x)} \langle \mathcal{Q} \left(\mathcal{O} \Lambda^{\mathrm{UV}}_{\mu\nu}(x) \right) \rangle_{\mathrm{UV}} \delta g^{\mu\nu}(x) = 0. \tag{0.126}$$

Therefore, the expectation value of any $\mathcal{Q}$-closed, gauge invariant observable will be formally independent of the metric. The interesting, i.e. topological, observables of the theory are those that live in the cohomology associated to the differential $\mathcal{Q}$. Since $\mathcal{Q}$ squares to a gauge transformation as $\mathcal{Q}^2 = \delta_\phi$, we must restrict to gauge invariant observables. Indeed, what we are really doing is working in the equivariant cohomology of the space of gauge connections $\mathcal{A}(P)$ with respect to gauge transformations $\mathcal{G}$, denoted $H_{\mathcal{G}}(\mathcal{A}(P))$. In the next section we will fully explore this perspective.

At present, we will simply exhibit elements of $H_{\mathcal{G}}(\mathcal{A}(P))$. Consider the tower of densities

$$\mathcal{O}^{(0)} = \frac{1}{2}\mathrm{Tr}[\phi^2], \tag{0.127}$$

$$\mathcal{O}^{(1)} = -\mathrm{Tr}[\phi\psi_\mu]dx^\mu, \tag{0.128}$$

$$\mathcal{O}^{(2)} = \frac{1}{2}\mathrm{Tr}[\phi F_{\mu\nu} + \psi_\mu\psi_\nu]dx^\mu \wedge dx^\nu, \tag{0.129}$$

$$\mathcal{O}^{(3)} = -\frac{1}{2}\mathrm{Tr}[\psi_\mu F_{\nu\rho}]dx^\mu \wedge dx^\nu \wedge dx^\rho, \tag{0.130}$$

$$\mathcal{O}^{(4)} = \frac{1}{4}\mathrm{Tr}[F_{\mu\nu}F_{\rho\sigma}]dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma. \tag{0.131}$$

Under the action of the scalar $\mathcal{Q}$, they satisfy the *descent equation*

$$\mathcal{Q}\mathcal{O}^{(n)} = d\mathcal{Q}^{(n-1)}, \tag{0.132}$$

with d the exterior derivative of $\mathbb{X}$. Integrating these observable densities over appropriate cycles, we define the *n-observables* $\mathcal{O}(\Sigma_n)$ associated to the n-cycles Σ as

$$\mathcal{O}^{(n)}(\Sigma_n) = \int_{\Sigma_n} \mathcal{O}^{(n)}. \tag{0.133}$$

These classes only depend on the homology classes of Σ_n in cohomology, as if $\partial\Sigma_n = 0$, then

$$\mathcal{Q}\mathcal{O}^{(n)}(\Sigma_n) = \int_{\Sigma_n} d\mathcal{O}^{(n-1)} = \int_{\partial\Sigma_n} \mathcal{O}^{(n-1)} = 0, \tag{0.134}$$

via the descent equation. Likewise, if $\Sigma_n = \partial\Sigma_{n+1}$, then

$$\mathcal{O}^{(n)}(\Sigma_n) = \int_{\Sigma_{n+1}} d\mathcal{O}^{(n)} = \int_{\Sigma_{n+1}} \mathcal{Q}\mathcal{O}^{(n+1)} = \mathcal{Q}\mathcal{O}^{(n+1)}(\Sigma_{n+1}). \tag{0.135}$$

This is reminiscent of the Donaldson map μ_{D} in (0.40), where each $\Sigma_n \in H_n(\mathbb{X})$ is mapped to a cohomology class $H^{4-n}(\mathcal{M}_{k,g})$, where we recall that $\mathcal{M}_{k,g}$ is the subspace of $\mathcal{A}/\mathcal{G}$ where $F_A^+ = 0$. The correspondence is even more robust when we identify the $\mathfrak{u}(1)_{\text{R}}$ charge of our twisted vector multiplet fields as the cohomological degree of $H_{\mathcal{G}}(\mathcal{A})$, so that our n-observables has degree $4-n$.

The observation above this suggests that we should integrate our n-observables over $\mathcal{M}_{k,g}$ as we did for the Donaldson polynomial invariants. As it turns out, this is precisely what is down by the path integral in our twisted theory! To fully understand this, we turn to the *Mathai-Quillen formalism,* though we will work in a direction reverse to the standard approach. For those interested in pedagogy, we direct the reader to the wonderful references [3, 10].

Suppose we have a $\mathcal{Q}$-closed, gauge invariant observable $\mathcal{O}$. We then suggestively write its expectation value as

$$\begin{aligned}\langle\mathcal{O}\rangle_{\text{UV}} &= \int [d\text{VM}]\mathcal{O}e^{-S_{\text{UV}}} \\ &= \int_{\widehat{\mathcal{A}}}[dAd\psi]\int_{\widehat{\Omega_g^{2,+}(\mathbb{X})}}[dHd\chi]\int_{\mathsf{Lie}\mathcal{G}}[d\phi]\int_{\widehat{\mathsf{Lie}\mathcal{G}}}[d\lambda d\eta]\mathcal{O}e^{-\mathcal{Q}(V_{\text{VU}}^{\text{Loc}}+V_{\text{UV}}^{\text{Pro}}+V_{\text{UV}}^{\text{Pot}})-2\pi i\tau_0 k}\end{aligned} \tag{0.136}$$

where we have introduced a myriad of new definitions. Here, for reasons that will soon become clear, we have split V_{UV} into

$$V_{\text{UV}}^{\text{Loc}} = \frac{1}{g_0^2}\int_{\mathbb{X}} d^4x\sqrt{g}\text{Tr}\left[\left(F_{\mu\nu} - \frac{1}{2}H_{\mu\nu}\right)\chi_{\mu\nu}\right], \tag{0.137}$$

$$V_{\mathrm{UV}}^{\mathrm{Pro}} = -\frac{1}{g_0^2}\int_{\mathbb{X}} d^4x\sqrt{g}\mathrm{Tr}\left[2\lambda D_\mu \psi^\mu\right], \tag{0.138}$$
$$V_{\mathrm{UV}}^{\mathrm{Pot}} = -\frac{1}{g_0^2}\int_{\mathbb{X}} d^4x\sqrt{g}\mathrm{Tr}\left[2\eta[\phi,\lambda]\right] \tag{0.139}$$

Next, turning to the integration, we define $\widehat{M}$ of a manifold as the superspace of the tangent bundle, so that $\widehat{M} = \Pi TM$. With this in mind, we see that λ and η are the coordinates of $\widehat{\mathsf{Lie}\mathcal{G}}$, A_μ and ψ_μ of $\widehat{\mathcal{A}(P)}$, and $H_{\mu\nu}$ and $\chi_{\mu\nu}$ of $\widehat{\Omega_g^{2,+}(\mathbb{X})}$. Working generally, suppose that the even coordinates of $\widehat{M}$ are given by x and the odd by ψ. Integration over both x and ψ is isomorphic to integration over M with differential forms. Under this isomorphism, we map each odd ψ coordinate to a generator dx of T^*M. Thus, writing $\omega_{\mathcal{O}} \in \Omega^*(M)$ for the image of $\mathcal{O} \in C^\infty(\widehat{M})$, we have

$$\int_{\widehat{M}} [dxd\psi]\mathcal{O} = \int_M \omega_{\mathcal{O}}. \tag{0.140}$$

Returning to $\langle\mathcal{O}\rangle_{\mathrm{UV}}$, our goal is to realize that the right hand side of (0.136) is conducting an integration of $\omega_{\mathcal{O}}$ over $\mathcal{M}_{k,g}$. To begin, let us first deal with the quotient by gauge transformations, or, as we will refer to it, projection. Putting aside the issue of reducible connections for a moment, consider the principal $\mathcal{G}$ bundle $\pi : \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{G}$. We would like to write integrals over the base of this bundle as integrals on the total space. That is, for an element $\omega \in H^*_{\mathcal{G}}(\mathcal{A})$, we want

$$\int_{\mathcal{A}/\mathcal{G}} \omega = \int_{\mathcal{A}} \pi^*(\omega) P(\mathcal{A} \to \mathcal{A}/\mathcal{G}), \tag{0.141}$$

where $P(\mathcal{A} \to \mathcal{A}/\mathcal{G})$ is the *projection form.* As we will see later, the equivariant cohomology of $\omega \in H^*_{\mathcal{G}}(\mathcal{A})$ is actually generated by the fields A, ψ, *and* ϕ, so the correct statement is technically given by

$$\int_{\mathcal{A}/\mathcal{G}} \omega = \int_{\mathsf{Lie}\mathcal{G}} [d\phi] \int_{\mathcal{A}} \pi^*(\omega) P(\mathcal{A} \to \mathcal{A}/\mathcal{G}), \tag{0.142}$$

This is known as the integration of equivariant forms and was introduced in [86]. Further, a representative projection form is given exactly by the integral[28]

$$P(\mathcal{A} \to \mathcal{A}/\mathcal{G}) = \int_{\widehat{\mathsf{Lie}\mathcal{G}}} [d\lambda d\eta] e^{-\mathcal{Q}V_{\mathrm{UV}}^{\mathrm{Pro}}}. \tag{0.143}$$

which is identical to a term in $\langle \mathcal{O} \rangle_{\mathrm{UV}}$.

Next, we turn to the localization to solutions of $s(A) = F_A^+ = 0$. In general, given a vector bundle $\pi : E \longrightarrow M$, where the rank of E is m, there is an isomorphism between the cohomology of M and the cohomology of E with compact vertical support, that is $H^n(M) \cong H^{n+m}_{\mathrm{CVS}}(E)$ via $\omega \mapsto \pi^*(\omega) \wedge \Phi(E)$. This isomorphism is known as the *Thom isomorphism* and $\Phi(E)$ as the *Thom form* [7]. In physics, we prefer Gaussian decay, or rapid descent, as opposed to compact vertical support, which is easily implemented after adding the structure of a Riemannian metric on the fibres of E. Given a generic section s of E, if M is compact, the pullback of $\Phi(E)$ is Poncaré dual to the zero locus of s, denoted $\mathcal{Z}(s)$, so we have

$$\int_M \omega \wedge s^*(\Phi(E)) = \int_{\mathcal{Z}(s)} \iota^*(\omega), \tag{0.144}$$

where $\iota : \mathcal{Z}(s) \hookrightarrow M$ is the inclusion map. From this, we see that $s^*(\Phi(E))$ is a representative of the *Euler class* of E, denoted $\mathrm{Eul}(E)$ or $\mathrm{Eul}_s(E, \boldsymbol{\nabla})$ when a section s or connection $\boldsymbol{\nabla}$ is specified. The Euler class measures the "twistedness" of a bundle and measures (with sign) the number of intersections of a generic section with the

[28] Consult Section 14.3.3 of [10] for a concise proof.

zero section. Further, if one specifies a connection $\boldsymbol{\nabla}$ on E and s is not generic, the above relation is changed. In such cases $\mathrm{Coker}\boldsymbol{\nabla}s \neq 0$, and we will have

$$\int_M \omega \wedge s^*(\Phi(E)) = \int_{\mathcal{Z}(s)} \iota^*(\omega) \wedge \mathrm{Eul}(\mathrm{Coker}\boldsymbol{\nabla}s \to \mathcal{Z}(s)), \tag{0.145}$$

where the second form on the right hand side is the Euler class of the bundle $\mathrm{Coker}\boldsymbol{\nabla}s \to \mathcal{Z}(s)$. We will return to this caveat shortly when we reconsider reducible connections.

The Mathai-Quillen formalism gives an explicit construction of the Thom form as an element of equvariant cohomology that is normalized to one [65]. Taking the pullback of this universal Thom form through the desired section, one arrives at an integral representation of the Euler class $\mathrm{Eul}_s(E, \boldsymbol{\nabla})$. Without rifling through the details,[29] this story can be formally extended to the infinite case when the zero locus $s^{-1}(0)$ is finite dimensional. Thus, we can consider it for our particular case of $\mathcal{E} = \mathcal{A} \times \Omega^{2,+}_g(\mathbb{X}) \longrightarrow \mathcal{A}$ with section $s(A) = F^+_A$. Written in superspace, it takes the form

$$\widehat{\mathrm{Eul}}_s(\mathcal{E}, \boldsymbol{\nabla}) = \int_{\widehat{\Omega^{2,+}_g(\mathbb{X})}} [dH d\chi] e^{-\mathcal{Q}V^{\mathrm{Loc}}_{\mathrm{UV}}}, \tag{0.146}$$

and formally satisfies

$$\int_{\widehat{\mathcal{A}}} [dA d\psi] \mathcal{O} \widehat{\mathrm{Eul}}_s(\mathcal{E}, \boldsymbol{\nabla}) = \int_{\mathcal{Z}(s)} \iota^*(\omega_{\mathcal{O}}). \tag{0.147}$$

Note that since $\mathcal{E}$ is a trivial bundle, $\mathrm{Coker}\boldsymbol{\nabla}s = 0$. This expression will soon be modified when we include projection.

[29] Physical mathematicians of the world unite, we have nothing to lose but extraneous details (though we know of them and realize their importance.)

And indeed, putting everything together, we have the grand result that

$$\int_{\widehat{\mathcal{E}}}[dAd\psi dHd\chi]\int_{\mathsf{Lie}\mathcal{G}}[d\phi]\int_{\widehat{\mathsf{Lie}\mathcal{G}}}[d\lambda d\eta]\mathcal{O}e^{-\mathcal{Q}(V^{\mathrm{Loc}}_{\mathrm{VU}}+V^{\mathrm{Pro}}_{\mathrm{UV}})}=\int_{\mathcal{M}_{g,k}}\omega_{\mathcal{O}}\wedge\mathrm{Eul}(\mathrm{Coker}\mathbb{F}),\tag{0.148}$$

where we recall $\mathbb{F}$ as the map $\boldsymbol{\nabla}\oplus D_A^\dagger$ from (0.34). Of course, generically away from reducible connections, the right hand side here reduces to a simple $\int_{\mathcal{M}_{g,k}}\omega_{\mathcal{O}}$, just as desired. In any case, it is clear that the integral is only non-vanishing when $\mathrm{deg}\omega_{\mathcal{O}}=\mathrm{Index}\mathbb{F}$.

Before moving on, let us note the two differences between $\langle\mathcal{O}\rangle_{\mathrm{UV}}$ in (0.136) and (0.148) above. The first is the $\exp[-\mathcal{Q}V^{\mathrm{Pot}}_{\mathrm{UV}}]$ which is clearly $\mathcal{Q}$-exact and should not ruffle our feathers given our earlier discussion. Further, it contains a scale potential $V=\frac{1}{2}[\phi,\lambda]^2$ whose solutions parameterize the classical vacua of the theory. The second term is $\exp[-2\pi i\tau_0 k]$ which is both $\mathcal{Q}$-closed and a topological invariant in its own right, so can happily come along for the ride.

Coming to the conclusion of this chapter of the story, let us return to the correlation functions of our n-observables. We define

$$\mathfrak{P}^{\ell,r}_{\mathrm{W}}(p,\Sigma)=\langle\big(\mathcal{O}^{(0)}(p)\big)^{\ell}\big(\mathcal{O}^{(2)}(\Sigma)\big)^{r}\rangle_{\mathrm{UV}}.\tag{0.149}$$

Recalling that the degree of an n-observables in $H_{\mathcal{G}}(\mathcal{A})$ is $4-n$, we see that $\mathfrak{P}^{\ell,r}_{\mathrm{W}}$ vanishes unless $4\ell+2r=\mathrm{Index}\mathbb{F}$ " = " $\dim\mathcal{M}_{g,k}$. In addition, as the correlation functions of $\mathcal{Q}$-closed, gauge invariant objects, they will be formally independent of the metric. Having run the gauntlet through the Mathai-Quillen construction, it should now come as no surprise that, up to a constant prefactor, we have

$$\mathfrak{P}^{\ell,r}_{\mathrm{W}}(p,\Sigma)=\mathfrak{P}^{\ell,r}_{\mathrm{D}}(p,\Sigma) \tag{0.150}$$

as well as

$$\mathsf{Z}_{\mathrm{W}}[g,p_0,\Sigma]=\langle e^{\mathcal{O}^{(0)}(p_0)+\mathcal{O}^{(2)}(\Sigma)}\rangle_{\mathrm{UV}}=\mathsf{Z}_{\mathrm{D}}[g,p_0,\Sigma] \tag{0.151}$$

Having unified the perspectives, we thus refine our notation refer to both sides above as the *Donaldson-Witten partition function* Z_{DW}.

While outside the purview of this work, in order to compute Z_{DW} one can flow into the low energy regime and turn the crank of Seiberg-Witten theory. This flow is conducted by taking the length scales to infinity, which is the same as changing the metric. Since the twisted theory is topological, the flow is exact, that is, the UV invariants are exactly equal to the IR invariants. We can therefore write

$$\mathsf{Z}_{\mathrm{DW}}[g,p,\Sigma]=\langle e^{\mathcal{O}^{(0)}(p)+\mathcal{O}^{(2)}(\Sigma)}\rangle_{\mathrm{UV}}=\langle e^{\mathcal{O}^{(0)}_{\mathrm{IR}}(p)+\mathcal{O}^{(2)}_{\mathrm{IR}}(\Sigma)+\Sigma^2 T(u)}\rangle_{\mathrm{IR}}, \tag{0.152}$$

where the correlation function on the right hand side is the path integral weighted by the IR action S_{IR}. The term $T(u)$ is the so-called contact term which result from the surface Σ having self-intersection. The benefit of the IR theory is that it is abelian, which greatly simplifies the integral. Here, we have certainly been glib and glossed over a sea of complexities, so we refer the curious reader to [71, 87, 88].

But enough of the past and onto the future!

1 The Algebra

1.1 Overview

Having settled the features of the original Donaldson-Witten invariants, we now shift to our construction of the family invariants. An early idea from Donaldson [24] and later refined by Moore and Witten [71] was to consider Z_{DW} as a degree zero element of the equivariant cohomology of the space of metrics $\mathsf{Met}(\mathbb{X})$ with respect to orientation preserving diffeomorphisms $\mathsf{Diff}_+(\mathbb{X})$. This immediately implies the possibility of higher degree elements, which could be understood as diffeomorphism invariant differential forms on $\mathsf{Met}(\mathbb{X})$. Such forms could then be integrated over appropriate families of metrics to construct new smooth invariants of $\mathbb{X}$. These are the family invariants.

Recall that Z_{DW} was constructed by integrating elements of $H_{\mathcal{G}}(\mathcal{A}(P))$ over all twisted multiplet fields, which we should now view as a projection from a total space down to the base space of $H_{\mathsf{Diff}_+(\mathbb{X})}(\mathsf{Met}(\mathbb{X}))$. Therefore, extending to higher degrees in the base requires a similar extension in the total space. Prior to specialization to cohomology, this total space needs to include $\mathbb{M} = \mathsf{Met}(\mathbb{X}) \times \mathcal{A}$ and, to work equivariantly, we need to take the quotient by $\mathbb{G} = \mathcal{G} \rtimes \mathsf{Diff}_+(\mathbb{X})$. Therefore, our goal will be to understand the cohomology of "$\mathbb{M}/\mathbb{G}$." Since there are fixed points of $\mathbb{M}$ under $\mathbb{G}$, such as isometries of $\mathbb{X}$ and reducible connections of $\mathcal{A}$, this space is not a manifold, and it is difficult to speak of its cohomology in a smooth way. Therefore, one introduces the space

$$E\mathbb{G} \times_{\mathbb{G}} \mathbb{M} = (E\mathbb{G} \times \mathbb{M})/\mathbb{G}, \tag{1.1}$$

where $E\mathbb{G}$ is the total space of the *universal* $\mathbb{G}$*-bundle*, defined to be a contractible space equipped with a free action of $\mathbb{G}$. Here, the quotiented $\mathbb{G}$-action acts as $g \cdot (h,x) = (hg^{-1}, gx)$ for $g \in \mathbb{G}, h \in E\mathbb{G}$, and $x \in \mathbb{M}$. Since $E\mathbb{G}$ is contractible, it adds no new homotopy classes to $E\mathbb{G} \times \mathbb{M}$, and, since it has a free action, the quotient by $\mathbb{G}$ produces a well-defined manifold. This leads us to define the *equivariant cohomology* of $\mathbb{M}$ with respect to $\mathbb{G}$ as

$$H_{\mathbb{G}}(\mathbb{M}) = H(E\mathbb{G} \times_{\mathbb{G}} \mathbb{M}). \tag{1.2}$$

This construction can likewise be done arrive at the equivariant cohomologies of $H_{\mathcal{G}}(\mathcal{A}(P))$ and $H_{\mathsf{Diff}_+(\mathbb{X})}(\mathsf{Met}(\mathbb{X}))$. The maneuver around the singularities of "$\mathbb{M}/\mathbb{G}$" is known as the *Borel construction* and is, in general, not immediately tractable. What we desire is an algebraic construction, which will lead us to the *Cartan model*.

Before turning to our explicit construction of the Cartan model of $H_{\mathbb{G}}(\mathbb{M})$, we briefly recall the well known structure of $H_{\mathcal{G}}(\mathcal{A}(P))$ and $H_{\mathsf{Diff}_+(\mathbb{X})}(\mathsf{Met}(\mathbb{X}))$.

1.2 $H_{\mathcal{G}}(\mathcal{A}(P))$

Recall the transformation laws of the $\mathcal{N} = 2$ twisted vector multiplet from (0.82)-(0.88). As we have previous alluded, the transformations of (A, ψ, ϕ) are a presentation of the *base* of the Cartan model of equivariant cohomology of $\mathcal{A}(P)$ with respect to $\mathcal{G}$ where the scalar supercharge $\mathcal{Q}$ plays the role of the differential. In addition, (λ, η) and $(\chi, D/H)$ form modules for $H_{\mathcal{G}}(\mathcal{A}(P))$ often called *anti-ghost multiplets*. The total complex for this model is given by

$$(\Omega^*(\mathcal{A}(P)) \otimes S^*(\mathsf{Lie}\mathcal{G}) \otimes \Omega^{2,+}_g(\mathbb{X}, \mathsf{ad}\, P) \otimes \Pi\Omega^{2,+}_g(\mathbb{X}, \mathsf{ad}\, P) \otimes \Omega^0(\mathbb{X}, \mathsf{ad}\, P) \otimes \Pi\Omega^0(\mathbb{X}, \mathsf{ad}\, P))^{\mathcal{G}}, \tag{1.3}$$

The superscript $\mathcal{G}$ denotes the fact that we are restricting to the $\mathcal{G}$-invariant subcomplex and $S^*(\mathsf{Lie}\mathcal{G})$ is the symmetric algebra of $\mathsf{Lie}\mathcal{G}$. This complex is only slightly altered from the presentation of Table 2, with the one difference that we now write $S^*(\mathsf{Lie}\mathcal{G})$ for the home of the scalar ϕ. We also now refer to the $\mathfrak{u}(1)$ charge as the *gauge degree*, as summarized in the table below.

Field	Gauge Degree
A_μ	0
ψ_μ	1
ϕ	2
λ	-2
η	-1
$H_{\mu\nu}$	0
$\chi_{\mu\nu}$	-1

Table 3: The gauge degree of the fields of $H_{\mathcal{G}}(\mathcal{A}(P))$ and its modules.

For convenience, we repeat the transformation laws here, collected as

$$\mathcal{Q}A_\mu = \psi_\mu, \qquad \mathcal{Q}\psi_\mu = -D_\mu\phi, \tag{1.4}$$

$$\mathcal{Q}\phi = 0, \tag{1.5}$$

$$\mathcal{Q}\lambda = \eta, \qquad \mathcal{Q}\eta = [\phi, \lambda], \tag{1.6}$$

$$\mathcal{Q}\chi_{\mu\nu} = H_{\mu\nu}, \qquad \mathcal{Q}H_{\mu\nu} = [\phi, \chi_{\mu\nu}]. \tag{1.7}$$

Note that we have specialized to H over D, primarily due to the simplicity of its transformation law. We also recall that

$$\mathcal{Q}^2 = \delta_\phi, \tag{1.8}$$

where δ_ϕ is a left-action gauge transformation by ϕ. Since, our complex (1.3) is invariant under such gauge transformations, the differential closes as desired.

1.3 $H_{\mathsf{Diff}_+(\mathbb{X})}(\mathsf{Met}(\mathbb{X}))$

The other well known Cartan model of relevance is the equivariant cohomology of $\mathsf{Met}(\mathbb{X})$ with respect to $\mathsf{Diff}_+(\mathbb{X})$. Its complex is given by

$$\left(\Omega^*(\mathsf{Met}(\mathbb{X})) \otimes S^*(\mathsf{diff}(\mathbb{X}))\right)^{\mathsf{Diff}_+(\mathbb{X})}. \tag{1.9}$$

Again, the superscript of $\mathsf{Diff}_+(\mathbb{X})$ denotes the projection to the diffeomorphism invariant subcomplex. The first term is generated by the Riemannian metric g and the symmetric gravitinos Ψ. The second term in our complex is generated by the vector field Φ, which can be considered as a local diffeomorphism. In addition, each of these fields is equipped with a *gravity degree* as shown in the table below.

Field	Gravity Degree
$g_{\mu\nu}$	0
$\Psi_{\mu\nu}$	1
Φ^μ	2

Table 4: The gravity degree of the fields of $H_{\mathsf{Diff}_+(\mathbb{X})}(\mathsf{Met}(\mathbb{X}))$.

To avoid conflating the differential of this model with any others,[30] we elect to denote it by d. We have

$$\mathsf{d}g_{\mu\nu} = \Psi_{\mu\nu}, \qquad \mathsf{d}\Psi_{\mu\nu} = \nabla_\mu\Phi_\nu + \nabla_\nu\Phi_\mu, \tag{1.10}$$

$$\mathsf{d}\Phi^\mu = 0. \tag{1.11}$$

Note that

$$\mathsf{d}^2 = \mathcal{L}_\Phi, \tag{1.12}$$

[30] Letting typeface guide, each differential has been chosen to match the style of the associated Lie group and manifold. Thus we have `mathcal` for $H_{\mathcal{G}}(\mathcal{A})$, `mathsf` for $H_{\mathsf{Diff}_+(\mathbb{X})}(\mathsf{Met}(\mathbb{X}))$, and `mathbb` for $H_{\mathbb{G}}(\mathbb{M})$.

where $\mathcal{L}_\Phi$ is a Lie derivative of $\mathbb{X}$ along Φ. Hence, $\mathsf{d}^2 = 0$ on our invariant subcomplex.

Further, as we will be working frequently with this differential and those like it, we note that it is not blind to the raising and lower of spatial indices and obeys

$$\mathsf{d}g^{\mu\nu} = -\Psi^{\mu\nu}. \tag{1.13}$$

1.4 $H_{\mathbb{G}}(\mathbb{M})$

We will now construct the Cartan model of equivariant cohomology of $\mathbb{M}$ with respect to $\mathbb{G}$, thus providing a realization of $H_{\mathbb{G}}(\mathbb{M})$.[31] We shall do so in three steps, first constructing the Weil model, making a suitable choice of horizontal generator, and then conducting the Mathai-Quillen isomorphism to the Cartan model. Though our work will be entirely self-contained, we point the interested reader to the two texts [40] and [82] for more details.

The resulting algebra takes the form

$$\mathbb{Q}g_{\mu\nu} = \Psi_{\mu\nu}, \qquad \mathbb{Q}A_\mu = \psi_\mu, \tag{1.14}$$

$$\mathbb{Q}\Psi_{\mu\nu} = \nabla_\mu\Phi_\nu + \nabla_\nu\Phi_\mu, \qquad \mathbb{Q}\psi_\mu = -D_\mu\phi + \Phi^\sigma F_{\sigma\mu}, \tag{1.15}$$

$$\mathbb{Q}\Phi^\sigma = 0, \qquad \mathbb{Q}\phi = -\Phi^\sigma\psi_\sigma. \tag{1.16}$$

where $g \in \mathsf{Met}(\mathbb{X})$ is the metric, $\Psi \in \Omega^1(\mathsf{Met}(\mathbb{X}))$ is the symmetric gravitino, and $\Phi \in \mathrm{Vect}(\mathbb{X})$ is a vector field whose role and physical origin will be revealed in due time. All other fields are precisely as they were in the twisted vector multiplet.

Further, in order to fully generalize twist supersymmetry, we will also need to provide transformation laws of (χ, H, λ, η) under $\mathbb{Q}$. We will understand these inclusions as adding anti-ghost multiplets to the algebra, or precisely, modules for $H_{\mathbb{G}}(\mathbb{M})$.

[31] Sticklers will worry about the non-compactness of both our groups, so we point them to [38].

One of the most important subtleties in this story is the fact that our group $\mathbb{G}$ is a semi-direct product, *not* a direct product. This follows from the fact that local diffeomorphisms and gauge transformations do not commute when acting on the space of adjoint valued differential forms. To see this, consider the group $\mathsf{Diff}(\mathsf{ad}\, P)$ of diffeomorphisms of the adjoint bundle. We have $\mathsf{Diff}_+(\mathbb{X})$ as the subgroup of diffeomorphisms of the form

$$\varphi_f : (x, y) \longrightarrow (f(x), y), \tag{1.17}$$

where f is an orientation preserving diffeomorphism $f \in \mathsf{Diff}_+(\mathbb{X})$ and (x, y) is a local coordinate with $x \in \mathbb{X}$ and y is an element of the adjoint representation of G. For the trivial P bundle, we can understand $\mathcal{G}$ as $\mathrm{Map}(\mathbb{X}, G)$, so that an element $g \in \mathcal{G}$ is a family of diffeomorphisms parametrized by $\mathbb{X}$. Thus, for each $x \in \mathbb{X}$, we have a diffeomorphism of the fibre of $\mathsf{ad}\, P$, $g_x : y \to g(y; x)$. Thus, we have $\mathcal{G}$ as the subgroup of $\mathsf{Diff}(\mathsf{ad}\, P)$ of elements of the form

$$\varphi_g : (x, y) \longrightarrow (x, g(y; x)). \tag{1.18}$$

We then take $\mathbb{G}$ to be the subgroup of elements in both subgroups $\mathcal{G}$ and $\mathsf{Diff}_+(\mathbb{X})$. Here, $\mathsf{Diff}_+(\mathbb{X})$ acts as a group of automorphisms of $\mathcal{G}$ via

$$\begin{aligned}(\varphi_f \varphi_g \varphi_f^{-1})(x, y) &= (\varphi_f \varphi_g)(f^{-1}(x), y) \\ &= \varphi_f(f^{-1}(x), g(y; f^{-1}(x)))) \\ &= (x, g(y; f^{-1}(x))).\end{aligned} \tag{1.19}$$

Therefore, for any $g \in \mathcal{G}$ and $f \in \mathsf{Diff}_+(\mathbb{X})$, we have $\varphi_f \varphi_g \varphi_f^{-1} = \varphi_{g'}$, with $g' = f^*(g) \in \mathcal{G}$. Hence, we realize $\mathbb{G}$ as a semi-direct product. We will see the repercussions of this shortly.

1.4.1 Weil Model

To begin, let us introduce the Weil algebra, which models $E\mathbb{G}$. Its complex is given by[32]

$$\mathcal{W}(\mathrm{Lie}\,\mathbb{G}) = S^*(\mathrm{Lie}\,\mathbb{G}) \otimes \Lambda^*(\mathrm{Lie}\,\mathbb{G}). \tag{1.20}$$

We take a basis of generators $\{\varphi^{\mathbb{A}}\}$ for the symmetric algebra $S^*(\mathsf{Lie}\mathbb{G})$ and generators $\{\theta^{\mathbb{A}}\}$ for the exterior algebra $\Lambda^*(\mathsf{Lie}\,\mathbb{G})$. In addition, we assign $\varphi^{\mathbb{A}}$ degree two and $\theta^{\mathbb{A}}$ degree one. Here, the index $\mathbb{A}$ is a 2-tuple of multi-indices for both the algebra of local gauge transformations and the algebra of local diffeomorphisms. Explicitly, one may write $\mathbb{A} = ((\vec{n}, a), (\vec{n}', \mu))$, where $\vec{n}$ and $\vec{n}'$ are indices for a basis of square-integrable functions on $\mathbb{X}$, a run over a basis of the Lie algebra $\mathfrak{g}$, and μ run over spatial indices i.e., $\mu = 1, 2, 3, 4$. We will typically ignore the square integrable functions basis, and thus write $\mathbb{A} = (a, \mu)$, where there is an implicit spatial coordinate dependence on any field carrying an $\mathbb{A}$ index.

The generators $\{\varphi^{\mathbb{A}}\}$ and $\{\theta^{\mathbb{A}}\}$ model the curvature and connection of the universal bundle $E\mathbb{G} \longrightarrow B\mathbb{G}$, where $B\mathbb{G} = E\mathbb{G}/\mathbb{G}$ is the *classifying space* of $\mathbb{G}$. Even though $E\mathbb{G}$ is topologically trivial, it is not a trivial bundle over $B\mathbb{G}$. Indeed, every $\mathbb{G}$ bundle, say $P \to M$, is a pullback of a *classifying map* $f : M \longrightarrow B\mathbb{G}$.

In this basis, we can realize the degree one Weil differential $d_{\mathcal{W}}$ as

$$d_{\mathcal{W}}\theta^{\mathbb{A}} = \varphi^{\mathbb{A}} - \frac{1}{2}[[\theta, \theta]]^{\mathbb{A}}, \qquad d_{\mathcal{W}}\varphi^{\mathbb{A}} = -[[\theta, \varphi]]^{\mathbb{A}}, \tag{1.21}$$

[32] The Weil algebra presented here ignores the fact that it is the Koszul algebra of the *dual* of $\mathsf{Lie}\,\mathbb{G}$. In the specific case under discussion, we are free to make such a gloss as we have a Killing form on $\mathfrak{g}$, a metric on tangent vectors, and an integration over $\mathbb{X}$ with volume form $\sqrt{g}d^4x$ for $L^2(\mathbb{X})$.

where the double brackets indicate the Lie bracket of $\mathsf{Lie}\mathbb{G}$. Alternatively, we can appeal to the structure constants of $\mathbb{G}$ and write

$$d_{\mathcal{W}}\theta^{\mathbb{A}} = \varphi^{\mathbb{A}} - \frac{1}{2} f_{\mathbb{B}\mathbb{C}}{}^{\mathbb{A}}\theta^{\mathbb{B}}\theta^{\mathbb{C}}, \qquad d_{\mathcal{W}}\varphi^{\mathbb{A}} = -f_{\mathbb{B}\mathbb{C}}{}^{\mathbb{A}}\theta^{\mathbb{B}}\varphi^{\mathbb{C}}, \tag{1.22}$$

where contracted indices are summed over. The first of these equations is Cartan's structure equation for $E\mathbb{G}$, as should be expected for a curvature and connection. Note that $d_{\mathcal{W}}^2 = 0$, so we can consider the cohomology $H^*(\mathcal{W}(\mathrm{Lie}\,\mathbb{G}))$. It is of course trivial, which is unsurprising since it was constructed to model $E\mathbb{G}$, a contractible space.

Additionally, the Weil algebra is equipped with degree -1 differential operators $I_{\mathbb{A}}$ and degree zero operators $L_{\mathbb{A}}$. They act as

$$I_{\mathbb{A}}\theta^{\mathbb{B}} = \delta_{\mathbb{A}}^{\mathbb{B}}, \qquad I_{\mathbb{A}}\varphi^{\mathbb{B}} = 0, \tag{1.23}$$

and

$$L_{\mathbb{A}}\theta^{\mathbb{B}} = -f_{\mathbb{A}\mathbb{C}}{}^{\mathbb{B}}\theta^{\mathbb{C}}, \qquad L_{\mathbb{A}}\varphi^{\mathbb{B}} = -f_{\mathbb{A}\mathbb{C}}{}^{\mathbb{B}}\varphi^{\mathbb{C}}. \tag{1.24}$$

While unimportant for our construction, $d_{\mathcal{W}}$, $I_{\mathbb{A}}$ and $L_{\mathbb{A}}$ form a Lie superalgebra. One relation of this Lie superalgebra is

$$L_{\mathbb{A}} = I_{\mathbb{A}}d_{\mathcal{W}} + d_{\mathcal{W}}I_{\mathbb{A}}, \tag{1.25}$$

which may also be taken as a definition of $L_{\mathbb{A}}$. Indeed, we call $I_{\mathbb{A}}$ the interior derivative of the Weil algebra, and $L_{\mathbb{A}}$ the Lie derivative of the Weil algebra. Note that $L_{\mathbb{A}}$ gives the (co)-adjoint action of $\mathsf{Lie}\mathbb{G}$ on itself, and therefore the Lie superalgebra relation

$$d_{\mathcal{W}} L_{\mathbb{A}} - L_{\mathbb{A}} d_{\mathcal{W}} = 0, \tag{1.26}$$

is equivalent to the statement that $d_{\mathcal{W}}$ is $\mathbb{G}$-equivariant.

In our construction, it is illuminating to split our basis according to the semi-direct product of $\mathbb{G} = \mathcal{G} \rtimes \mathsf{Diff}_{+}(\mathbb{X})$ (for the general case of the semi-direct product of two finite dimensional Lie groups we refer the reader to Appendix B). The Weil algebra's complex then splits as

$$\begin{aligned} \mathcal{W}(\mathsf{Lie}\mathbb{G}) &= S^{*}(\mathsf{Lie}\mathcal{G} \oplus \mathsf{diff}(\mathbb{X})) \otimes \Lambda^{*}(\mathsf{Lie}\mathcal{G} \oplus \mathsf{diff}(\mathbb{X})) \\ &= S^{*}(\mathsf{Lie}\mathcal{G}) \otimes \Lambda^{*}(\mathsf{Lie}\mathcal{G}) \otimes S^{*}(\mathsf{diff}(\mathbb{X})) \otimes \Lambda^{*}(\mathsf{diff}(\mathbb{X})) \\ &= \mathcal{W}(\mathsf{Lie}\mathcal{G}) \otimes \mathcal{W}(\mathsf{diff}(\mathbb{X})). \end{aligned} \tag{1.27}$$

With this splitting, we make definitions for the generators of each of the tensorands above, writing

$$\theta^{(a,0)} = \widetilde{c}^{a} \otimes 1, \qquad \varphi^{(a,0)} = \widetilde{\phi}^{a} \otimes 1, \tag{1.28}$$

$$\theta^{(0,\mu)} = 1 \otimes \xi^{\mu}, \qquad \varphi^{(0,\mu)} = 1 \otimes \Phi^{\mu}. \tag{1.29}$$

We will often write these fields without the tensored identity in what follows.

As explained above, $\mathbb{G}$ is a semi-direct product. On account of this, while the complex $\mathcal{W}(\mathsf{Lie}\mathbb{G})$ splits into a tensor product of the Weil complexes of the factors of $\mathbb{G}$, the actual algebra does not. This follows from the definition of the Lie bracket of $\mathsf{Lie}\,\mathbb{G}$ in (1.21) or, alternatively, from the form of the structure constants in (1.22).[33]

[33]For the correct treatment with the structure constants, we point to Appendix C, where the simple example of $\mathbb{X}$ as a four torus and $G = \mathrm{SU}(2)$ is considered. The crux of the issue is that the semi-direct product requires the structure constants to have dependence on the derivative of elements of $\mathsf{Lie}\,\mathcal{G}$, seemingly leading to structure functions. We see this in the second and third term

For $\epsilon_1, \epsilon_2 \in \mathrm{Lie}\,\mathcal{G}$ and $\eta_1, \eta_2 \in \mathsf{diff}(\mathbb{X})$, we have

$$\begin{aligned}[][(\epsilon_1,\eta_1),(\epsilon_2,\eta_2)]]^{(a,\mu)} &= ([\epsilon_1,\epsilon_2]^a + \rho_{\eta_1}(\epsilon_2)^a - \rho_{\eta_2}(\epsilon_1)^a,\ [\eta_1,\eta_2]^\mu) \\ &= ([\epsilon_1,\epsilon_2]^a + \eta_1^\sigma\partial_\sigma\epsilon_2^a - \eta_2^\sigma\partial_\sigma\epsilon_1^a, \eta_1^\sigma\partial_\sigma\eta_2^\mu - \eta_2^\sigma\partial_\sigma\eta_1^\mu)\,,\end{aligned} \tag{1.30}$$

where $\rho_\eta \in \mathrm{Aut}(\mathsf{Lie}\mathcal{G})$, $[\cdot,\cdot]^a$ is the Lie bracket of $\mathsf{Lie}\mathcal{G}$, and $[\cdot,\cdot]^\mu$ is the Lie bracket of $\mathsf{diff}(\mathbb{X})$. Thus we may write the action of $d_\mathcal{W}$ in our split basis as

$$d_\mathcal{W}\widetilde{c}^a = \widetilde{\phi}^a - \frac{1}{2}[\widetilde{c},\widetilde{c}]^a - \xi^\sigma\partial_\sigma\widetilde{c}^a, \quad d_\mathcal{W}\widetilde{\phi}^a = -[\widetilde{c},\widetilde{\phi}]^a - \xi^\sigma\partial_\sigma\widetilde{\phi}^a + \Phi^\sigma\partial_\sigma\widetilde{c}^a, \tag{1.31}$$

$$d_\mathcal{W}\xi^\mu = \Phi^\mu - \xi^\sigma\partial_\sigma\xi^\mu, \qquad d_\mathcal{W}\Phi^\mu = -\xi^\sigma\partial_\sigma\Phi^\mu + \Phi^\sigma\partial_\sigma\xi^\mu. \tag{1.32}$$

The derivative terms in both equations of (1.31) would not have been present were $\mathbb{G}$ simply a direct product will play an important role in what follows.

We can also express the differential operators $I_\mathbb{A}$ and $L_\mathbb{A}$ in this split basis. Define

$$I_\epsilon = \epsilon^a I_{(a,0)}, \qquad L_\epsilon = \epsilon^a L_{(a,0)}, \tag{1.33}$$

$$I_\eta = \eta^\mu I_{(0,\mu)}, \qquad L_\eta = \eta^\mu L_{(0,\mu)}. \tag{1.34}$$

The interior derivative then acts on the generators as

$$I_\epsilon\widetilde{c}^a = \epsilon^a, \qquad I_\epsilon\widetilde{\phi}^a = 0, \tag{1.35}$$

$$I_\epsilon\xi^\mu = 0, \qquad I_\epsilon\Phi^\mu = 0, \tag{1.36}$$

and

$$I_\eta\widetilde{c}^a = 0, \qquad I_\eta\widetilde{\phi}^a = 0, \tag{1.37}$$

of the first entry on the right hand side of (1.30). This is clear when one has a basis of Fourier modes, where the structure constants can be made explicit, as in section Appendix C.3.

$$I_\eta \xi^\mu = \eta^\mu, \qquad I_\eta \Phi^\mu = 0, \tag{1.38}$$

and the Lie derivative as

$$L_\epsilon \widetilde{c}^a = -[\epsilon, \widetilde{c}]^a + (-1)^{\deg(\epsilon)} \xi^\sigma \partial_\sigma \epsilon^a, \quad L_\epsilon \widetilde{\phi}^a = -[\epsilon, \widetilde{\phi}]^a + \Phi^\sigma \partial_\sigma \epsilon^a, \tag{1.39}$$

$$L_\epsilon \xi^\nu = 0 \;, \qquad L_\epsilon \Phi^\nu = 0, \tag{1.40}$$

and

$$L_\eta \widetilde{c}^a = -\eta^\sigma \partial_\sigma \widetilde{c}^a, \qquad L_\eta \widetilde{\phi}^a = -\eta^\sigma \partial_\sigma \widetilde{\phi}^a, \tag{1.41}$$

$$L_\eta \xi^\nu = -\eta^\sigma \partial_\sigma \xi^\nu + (-1)^{\deg(\eta)} \xi^\sigma \partial_\sigma \eta^\nu, \quad L_\eta \Phi^\nu = -\eta^\sigma \partial_\sigma \Phi^\nu + \Phi^\mu \partial_\mu \eta^\nu. \tag{1.42}$$

Of particular note is the inhomogenous (co)-adjoint action of gauge transformations on the generators $\widetilde{c}$ and $\widetilde{\phi}$ in (1.39).

The full complex for the Weil model is built from the tensor product of the Weil algebra and the de Rham complex of the manifold over which we are building our equivariant cohomology. In our case, that means we want to understand the space

$$\Omega^*(\mathbb{M}) = \Omega^*(\mathcal{A}) \otimes \Omega^*(\mathsf{Met}(\mathbb{X})). \tag{1.43}$$

The first factor is generated by gauge connections A and $d_\mathcal{A} A = \psi$ and the second by metrics g and $d_{\mathsf{Met}} g = \Psi$, where $d_\mathcal{A}$ and d_{Met} are the usual exterior differentials on the respective spaces. We introduce a single differential operator for this complex as

$$d_\mathbb{M} = d_\mathcal{A} \otimes 1 + 1 \otimes d_{\mathsf{Met}}, \tag{1.44}$$

so that we may summarize the algebra as

$$d_{\mathbb{M}} A_\mu = \psi_\mu, \qquad d_{\mathbb{M}} \psi_\mu = 0, \tag{1.45}$$

$$d_{\mathbb{M}} g_{\mu\nu} = \Psi_{\mu\nu}, \qquad d_{\mathbb{M}} \Psi_{\mu\nu} = 0. \tag{1.46}$$

This complex is also equipped with interior derivatives and Lie derivatives associated to the (co)-adjoint action of gauge transformations and diffeomorphisms on $\mathbb{M}$. Using the same notation as the previous section, let us define

$$I_\epsilon = \epsilon^a I_{X_a}, \qquad L_\epsilon = \epsilon^a L_{X_a}, \tag{1.47}$$

$$I_\eta = \eta^\mu I_{X_\mu}, \qquad L_\eta = \eta^\mu L_{X_\mu}, \tag{1.48}$$

where X_a and X_μ are vector fields on $\mathbb{M}$ generating the gauge transformations or diffeomorphisms associated to their respective indices. We then have the interior derivatives on $\mathbb{M}$ as

$$I_\epsilon A_\mu = 0, \qquad I_\epsilon \psi_\mu = D_\mu \epsilon, \tag{1.49}$$

$$I_\epsilon g_{\mu\nu} = 0, \qquad I_\epsilon \Psi_{\mu\nu} = 0, \tag{1.50}$$

and

$$I_\eta A_\mu = 0, \qquad I_\eta \psi_\mu = -\eta^\sigma (\nabla_\sigma A_\mu) - (\nabla_\mu \eta^\sigma) A_\sigma, \tag{1.51}$$

$$I_\eta g_{\mu\nu} = 0, \qquad I_\eta \Psi_{\mu\nu} = -\nabla_\mu \eta_\nu - \nabla_\nu \eta_\mu. \tag{1.52}$$

The Lie derivatives on $\mathbb{M}$ are given by

$$L_\epsilon A_\mu = D_\mu \epsilon, \qquad L_\epsilon \psi_\mu = -[\epsilon, \psi_\mu], \tag{1.53}$$

$$L_\epsilon g_{\mu\nu} = 0, \qquad L_\epsilon \Psi_{\mu\nu} = 0, \tag{1.54}$$

and

$$L_\eta A_\mu = -(\eta^\sigma \nabla_\sigma) A_\mu - (\nabla_\mu \eta^\sigma) A_\sigma, \tag{1.55}$$

$$L_\eta \psi_\mu = -(\eta^\sigma \nabla_\sigma) \psi_\mu - (\nabla_\mu \eta^\sigma) \psi_\sigma, \tag{1.56}$$

$$L_\eta g_{\mu\nu} = -\nabla_\mu \eta_\nu - \nabla_\nu \eta_\mu, \tag{1.57}$$

$$L_\eta \Psi_{\mu\nu} = -\eta^\sigma (\nabla_\sigma \Psi_{\mu\nu}) - (\nabla_\mu \eta^\sigma) \Psi_{\sigma\nu} - (\nabla_\nu \eta^\sigma) \Psi_{\mu\sigma}. \tag{1.58}$$

We pause here to reivew our notation. We use ∇ to denote a metric covariant derivative which is metric compatible, that is, it satisfies $\nabla g = 0$. The derivative D (or D_A when we speak with form language) will be used to denote a gauge *and* metric compatible derivative, such that its action on any adjoint valued field ω can be expressed as $D_A \omega = \nabla \omega + [A, \omega]$.

As in the case of the Weil algebra, the Lie derivatives on $\mathbb{M}$ produce the co-adjoint action of $\mathbb{G}$. We can express this as

$$L_\epsilon = -\delta_\epsilon \quad \text{and} \quad L_\eta = -\mathcal{L}_\eta. \tag{1.59}$$

In order to keep our differential operators straight, we adhere to the rule the the calligraphic $\mathcal{L}$ will always refer to Lie derivatives along vectors fields on $\mathbb{X}$.

With the subcomplexes and the differential operators understood, we are ready to construct the Weil model of $\mathbb{G}$-equivariant cohomology of $\mathbb{M}$. We begin with the total Weil complex as

$$\Omega(\mathbb{M}) \otimes \mathcal{W}(\mathsf{Lie}\mathbb{G}), \tag{1.60}$$

which models $E\mathbb{G} \times \mathbb{M}$. Our total differential is the sum of each of the factors of the total complex, which we write as

$$d_{\mathrm{T}} = 1 \otimes d_{\mathcal{W}} + d_{\mathbb{M}} \otimes 1. \tag{1.61}$$

On its own, $H(\Omega(\mathbb{M}) \otimes \mathcal{W}(\mathsf{Lie}\,\mathbb{G}), d_{\mathrm{T}})$ is not the cohomology we are after, since we have not introduced an algebraic analogue to the quotient by $\mathbb{G}$. To do so, we need to restrict to the so called *basic classes*.

A basic class is an element $\omega \in \Omega(\mathbb{M}) \otimes \mathcal{W}(\mathsf{Lie}\,\mathbb{G})$ that is both *horizontal* and *invariant*. Horizontal means that

$$(1 \otimes I_{\mathbb{A}} + I_{X_{\mathbb{A}} \otimes 1})\omega = 0, \tag{1.62}$$

for all indices $\mathbb{A}$, and invariant means that

$$(1 \otimes L_{\mathbb{A}} + L_{X_{\mathbb{A}} \otimes 1})\omega = 0, \tag{1.63}$$

for all $\mathbb{A}$. We can understand basic classes as those that have no vertical components (horizontal) and no vertical variation (invariant), where vertical is understood to be the various directions of the group action of $\mathbb{G}$.

With our full complex and differential in hand, we have the equivariant de Rham theorem giving us the marvelous isomorphism

$$H_{\mathbb{G}}(\mathbb{M}) \cong H((\Omega(\mathbb{M}) \otimes \mathcal{W}(\mathsf{Lie}\,\mathbb{G}))_{\mathrm{basic}}, d_{\mathrm{T}}), \tag{1.64}$$

where we have restricted to the basic classes of our total Weil complex.

Before we present the action of the Weil differential, we must address an issue with our fields. Typically, in equivariant cohomology, we take φ to be the horizontal generator of the Weil algebra. In our case, φ splits into Φ and $\widetilde{\phi}$, which are both horizontal generators, but, as indicated in (1.39), $\widetilde{\phi}$ does *not* transform homogenously under gauge transformations. When we identify our model of equivariant cohomology as a background of twisted supergravity, we will want the bottom component of the vector multiplet to transform as a healthy adjoint valued scalar field. This is the field we wish to identify with $\widetilde{\phi}$. Thus we have to ask, how can we cure $\widetilde{\phi}$ of its inhomogeneity?

We answer the question by considering a shift of

$$\widetilde{\phi} \longrightarrow \phi = \widetilde{\phi} - \Phi^\sigma A_\sigma. \tag{1.65}$$

Since $\widetilde{\phi}, \Phi$, and A are all horizontal, so is ϕ, which will assist us shortly in our move into the Cartan model. Further, we have

$$\begin{aligned}(L_\epsilon \otimes 1 + 1 \otimes L_\epsilon)\phi^a &= -[\epsilon, \widetilde{\phi}^a] + \Phi^\sigma \partial_\sigma \epsilon^a - \Phi^\sigma(\partial_\sigma \epsilon^a + [A_\sigma, \epsilon]^a) \\ &= -[\epsilon, \widetilde{\phi} - \Phi^\sigma A_\sigma] = -[\epsilon, \phi].\end{aligned} \tag{1.66}$$

Thus we see that ϕ transforms homogenously under the co-adjoint action of gauge transformations! As one may investigate in Appendix B, equivariant cohomology with a semi-direct product group always leads to an inhomogenous action on the horizontal generator of the normal subgroup half of the Weil algebra. It is not at all typical that we can conduct a curing shift. The case at hand is special, since $\mathcal{A}$ is an affine space and the gauge connection A transforms in precisely the correct way to compensate for the mixed transformation of $\widetilde{\phi}$. This is a feature of equivariant cohomology on the space of connections of any principal bundles with respect to the semi-direct product of the group of diffeomorphisms of the base with the group of fibre preserving automorphisms.

We can also shift $\widetilde{c}$ to transform homogeneous, and do so by taking

$$\widetilde{c} \longrightarrow c = \widetilde{c} - \xi^\sigma A_\sigma. \tag{1.67}$$

Hence, we have the full algebra of the (shifted) Weil model as

$$d_{\mathrm{T}} g_{\mu\nu} = \Psi_{\mu\nu}, \tag{1.68}$$

$$d_{\mathrm{T}} A_\mu = \psi_\mu, \tag{1.69}$$

$$d_{\mathrm{T}} \Psi_{\mu\nu} = 0, \tag{1.70}$$

$$d_{\mathrm{T}} \psi_\mu = 0, \tag{1.71}$$

$$d_{\mathrm{T}} \Phi^\mu = -\xi^\sigma \nabla_\sigma \Phi^\mu + \Phi^\sigma \nabla_\sigma \xi^\mu, \tag{1.72}$$

$$d_{\mathrm{T}} \phi = -[c,\phi] - \xi^\sigma D_\sigma \phi + \Phi^\sigma D_\sigma c - \xi^\sigma \Phi^\rho F_{\sigma\rho} - \Phi^\sigma \psi_\sigma, \tag{1.73}$$

$$d_{\mathrm{T}} \xi^\mu = \Phi^\mu - \xi^\sigma \nabla_\sigma \xi, \tag{1.74}$$

$$d_{\mathrm{T}} c = \phi - \frac{1}{2}[c,c] - \xi^\sigma D_\sigma c - \frac{1}{2}\xi^\sigma \xi^\rho F_{\sigma\rho} + \xi^\sigma \psi_\sigma, , \tag{1.75}$$

where $d_{\mathrm{T}}^2 = 0$ and equivariant classes are restricted to the basic subcomplex i.e., satisfy (1.62) and (1.63).

1.4.2 Cartan Model

While working with a strictly nilpotent differential has its benefits, it is possible to trade this feature for an algebraic solution to the horizontal constraint of (1.62). We do so in two steps. First we conduct the Mathai-Quillen isomorphism which brings us to the so-called BRST, or intermediate model. Next, ignoring all vertical fields, we project onto the invariant subcomplex, to arrrive at the Cartan model.

Turning to the Mathai-Quillen isomorphism, define the operator

$$\gamma = I_{X_a} \otimes \widetilde{c}^a + I_{X_\mu} \otimes \xi^\mu, \tag{1.76}$$

factoring with respect to the total complex as in (1.60). This operator only acts non-trivially on the space $\Omega(\mathbb{M})$, and among the generators, only on Ψ and ψ. Moreover, exponentiating γ and, using it to conjugateour differential operators, we find

$$e^\gamma(1 \otimes I_{\mathbb{A}} + I_{X_{\mathbb{A}}} \otimes 1)e^{-\gamma} = 1 \otimes I_{\mathbb{A}}, \tag{1.77}$$

$$e^\gamma(1 \otimes L_{\mathbb{A}} + L_{X_{\mathbb{A}}} \otimes 1)e^{-\gamma} = (1 \otimes L_{\mathbb{A}} + L_{X_{\mathbb{A}}} \otimes 1). \tag{1.78}$$

This indicates that conjugating by e^γ solves the horizontal condition on $\Omega(\mathbb{M})$. We define the new differential

$$d_{\mathcal{C}} = e^\gamma d_{\mathrm{T}} e^{-\gamma}, \tag{1.79}$$

which we call the Cartan differential. In the given form, the differential inherits the nilpotency of d_{T}. Acting on the fields of the Weil model, we obtain

$$d_{\mathcal{C}} g_{\mu\nu} = \Psi_{\mu\nu} - \nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu, \tag{1.80}$$

$$d_{\mathcal{C}} \Psi_{\mu\nu} = \nabla_\mu \Phi_\nu + \nabla_\nu \Phi_\mu - \xi^\sigma \nabla_\sigma \Psi_{\mu\nu} - (\nabla_\mu \xi^\sigma)\Psi_{\sigma\nu} - (\nabla_\nu \xi^\sigma)\Psi_{\mu\sigma}, \tag{1.81}$$

$$d_{\mathcal{C}} \Phi^\mu = -\xi^\sigma \nabla_\sigma \Phi^\mu + \Phi^\sigma \nabla_\sigma \xi^\mu, \tag{1.82}$$

$$d_{\mathcal{C}} \xi^\mu = \Phi^\mu - \xi^\sigma \nabla_\sigma \xi^\mu, \tag{1.83}$$

and

$$d_{\mathcal{C}} A_\mu = \psi_\mu + D_\mu c - \xi^\sigma F_{\sigma\mu}, \tag{1.84}$$

$$d_{\mathcal{C}} \psi_\mu = -D_\mu \phi + \Phi^\sigma F_{\sigma\mu} - [c, \psi_\mu] - \xi^\sigma D_\sigma \psi_\mu - (\nabla_\mu \xi^\sigma)\psi_\sigma, \tag{1.85}$$

$$d_{\mathcal{C}}\phi \quad = -\Phi^{\sigma}\psi_{\sigma} - [c,\phi] - \xi^{\sigma}D_{\sigma}\phi, \tag{1.86}$$

$$d_{\mathcal{C}}c \quad = \phi - \frac{1}{2}[c,c] + \frac{1}{2}\xi^{\sigma}\xi^{\rho}F_{\sigma\rho}. \tag{1.87}$$

We call this new algebra the *BRST model*. Its complex is given by

$$\left(\Omega(\mathbb{M}) \otimes (\mathcal{W}(\mathsf{Lie}\,\mathbb{G}))_{\text{horizontal}}\right)^{\mathbb{G}}, \tag{1.88}$$

namely, the invariant classes of the total Weil model complex, but now with only the need to restrict to horizontal classes on the Weil algebra tensorand.

Next, we can further restrict our complex by projecting to the horizontal subcomplex of the Weil algebra. The vertical fields are identically the fields c and ξ, and thus any class which contains them is not an element of our cohomology. Thus we need only consider

$$\left(\Omega^{*}(\mathbb{M}) \otimes S^{*}(\mathsf{Lie}\mathbb{G})\right)^{\mathbb{G}}. \tag{1.89}$$

Further, inspecting (1.80)-(1.87), we can make the following observation. Considering the particular Lie derivative

$$L_{\widetilde{c}+\xi} = \widetilde{c}^{a}(1 \otimes L_{(a,0)} + L_{X_a} \otimes 1) + \xi^{\mu}(1 \otimes L_{(0,\mu)} + L_{X_\mu} \otimes 1), \tag{1.90}$$

we have

$$L_{\widetilde{c}+\xi}g_{\mu\nu} \ = -\nabla_{\mu}\xi_{\nu} - \nabla_{\nu}\xi_{\mu}, \tag{1.91}$$

$$L_{\widetilde{c}+\xi}\Psi_{\mu\nu} = -\xi^{\sigma}\nabla_{\sigma}\Psi_{\mu\nu} - (\nabla_{\mu}\xi^{\sigma})\Psi_{\sigma\nu} - (\nabla_{\nu}\xi^{\sigma})\Psi_{\mu\sigma}, \tag{1.92}$$

$$L_{\widetilde{c}+\xi}\Phi^{\mu} \ = -\xi^{\sigma}\nabla_{\sigma}\Phi^{\mu} + \Phi^{\sigma}\nabla_{\sigma}\xi^{\mu}, \tag{1.93}$$

$$L_{\widetilde{c}+\xi}A_{\mu} \ = D_{\mu}c - \xi^{\sigma}F_{\sigma\mu}, \tag{1.94}$$

$$L_{\widetilde{c}+\xi}\psi_\mu = -[c,\psi_\mu] - \xi^\sigma D_\sigma\psi_\mu - (\nabla_\mu\xi^\sigma)\psi_\sigma, \tag{1.95}$$

$$L_{\widetilde{c}+\xi}\phi = -[c,\phi] - \xi^\sigma D_\sigma\phi. \tag{1.96}$$

These give exactly the last terms on the right hand sides of (1.80)-(1.82) and (1.84)-(1.86). Hence, when considered as elements inside the subcomplex (1.89), each of the above terms vanishes due to the invariant projection. Therefore we can freely project the action of our Cartan differential to this invariant subcomplex leading to the final algebra of the Cartan model. We elect to rename $d_{\mathcal{C}}$ to $\mathbb{Q}$ in order to distinguish it from the differential prior to the invariant projection. Hence, we arrive at

$$\mathbb{Q}g_{\mu\nu} = \Psi_{\mu\nu}, \qquad \mathbb{Q}A_\mu = \psi_\mu, \tag{1.97}$$

$$\mathbb{Q}\Psi_{\mu\nu} = \nabla_\mu\Phi_\nu + \nabla_\nu\Phi_\mu, \qquad \mathbb{Q}\psi_\mu = -D_\mu\phi + \Phi^\sigma F_{\sigma\mu}, \tag{1.98}$$

$$\mathbb{Q}\Phi^\sigma = 0, \qquad \mathbb{Q}\phi = -\Phi^\sigma\psi_\sigma. \tag{1.99}$$

The price of this projection is that we have lost the general nilpotency of $d_{\mathcal{C}}$ and instead have

$$\mathbb{Q}^2 = \delta_{\phi+\Phi^\sigma A_\sigma} + \mathcal{L}_\Phi = \delta_\phi + \mathcal{L}_\Phi^{(A)}, \tag{1.100}$$

where we recall in (1.8) that δ_ϕ is the right-action of a gauge transformation by ϕ, $\mathcal{L}_\Phi$ is the Lie derivative of the four manifold $\mathbb{X}$ along the vector field Φ, and we introduce the notation $\mathcal{L}_\Phi^{(A)}$ to indicate the gauge covariant Lie derivative of $\mathbb{X}$ along the vector field Φ, namely $\mathcal{L}_\Phi$ where we replace all metric covariant derivatives ∇ by the gauge and metric covariant derivatives D_A. This differential *is* nilpotent on the invariant subcomplex of (1.89).

All together, noting that the Mathai-Quillen isomorphism is actually a quasiisomorphism, we obtain the desired result of

$$H_{\mathbb{G}}(\mathbb{M}) \cong H((\Omega^*(\mathbb{M}) \otimes S^*(\mathsf{Lie}\,\mathbb{G}))^{\mathbb{G}}, \mathbb{Q}). \tag{1.101}$$

We note in passing that the transformation laws (1.97)-(1.99) have appeared previous in the work of [42] (see also the work [43] for a similar construction), though in a different context. In particular, their degree two field, denoted γ^μ is required to be a Killing vector field, whereas we do not assume that $\mathbb{X}$ has any isometries and allow Φ^μ to be any vector field. Further, in these works, γ^μ is constructed as a bilinear in Majorana ghosts of a supergravity theory, whereas, as we will see, our Φ^μ field emerges as a ghost field for the vector supersymmetry of a twisted and truncated theory of supergravity.

1.4.3 Anti-Ghost Multiplets

Having identified the Cartan model for the $\mathbb{G}$-equivariant cohomology of $\mathbb{M}$, we wish to include the other fields in the $\mathcal{N} = 2$ twisted vector multiplet. To do so we introduce the four additional fields from Table 2 which divide into what we call the *projection multiplet* (λ, η) and the *localization multiplet* (χ, H). These will be understood as two modules for $H_{\mathbb{G}}(\mathbb{M})$ and their names reflect their role in the Mathai-Quillen formalism.

Note that

$$\mathbb{Q}\big|_{\Psi,\Phi=0} = \mathcal{Q}, \tag{1.102}$$

on the Cartan base multiplet (A, ψ, ϕ). On the anti-ghost multiplets, the action of $\mathcal{Q}$ is introduced in (1.6) and (1.7) as a *contractible pair*, namely, defined so that $\mathcal{Q}$ on the lower degree field gives the higher degree field, and, upon acting on the higher degree field, is in agreement with $\mathcal{Q}^2 = \delta_\phi$. We wish to extend the action of $\mathcal{Q}$ to one of $\mathbb{Q}$ on the λ, η, χ and H fields such that it reproduces $\mathcal{Q}$ when $\Phi, \Psi = 0$, and that it maintain $\mathbb{Q}^2 = \delta_\phi + \mathcal{L}_\Phi^{(A)}$.

We first turn to the projection multiplet. In $H_{\mathcal{G}}(\mathcal{A}(P))$ we have

$$\mathcal{Q}\lambda = \eta, \qquad \mathcal{Q}\eta = [\phi, \lambda]. \tag{1.103}$$

where λ and η are adjoint valued scalar fields which are commuting and anti-commuting respectively. With such simple form, there is one obvious extension to $\mathbb{Q}$, namely

$$\mathbb{Q}\lambda = \eta, \qquad \mathbb{Q}\eta = [\phi, \lambda] + \Phi^\sigma D_\sigma \lambda. \tag{1.104}$$

Indeed, this is what we do, as it satisfies (1.100). Sometimes things are simple.

Sometimes thing are complicated. The localization multiplet of (1.7) gives

$$\mathcal{Q}\chi_{\mu\nu} = H_{\mu\nu}\ , \qquad \mathcal{Q}H_{\mu\nu} = [\phi, \chi_{\mu\nu}], \tag{1.105}$$

Note that both χ and H are self-dual fields, a condition that has explicit metric dependence. Hence our variation in the combined Cartan model requires a variation of the self-dual constraint. Following the methods of Appendix D, a minimal extension this action to $\mathbb{Q}$ is to take

$$\mathbb{Q}\chi_{\mu\nu} = H_{\mu\nu} - (\Psi^\sigma{}_{[\mu}\chi_{\nu]\sigma})^-, \tag{1.106}$$

where the second term on the right hand side is the required anti-self-dual part cf. (D.3)-(D.13). Next, we take $\mathbb{Q}H_{\mu\nu}$ to be whatever is necessary for the algebra to close. Hence, defining $\mathsf{B}_{\mu\nu} = \Psi^{\sigma}{}_{[\mu}\chi_{\nu]\sigma}$ and using (D.14), we compute

$$\begin{aligned}
\mathbb{Q}^2\chi_{\mu\nu} &= \mathbb{Q}H_{\mu\nu} - \mathbb{Q}(\mathsf{B}^-_{\mu\nu}) \\
&= \mathbb{Q}H_{\mu\nu} - (\mathbb{Q}\mathsf{B}_{\mu\nu})^- - (\Psi^{\sigma}{}_{[\mu}\mathsf{B}^+_{\nu]\sigma})^- + (\Psi^{\sigma}{}_{[\mu}\mathsf{B}^-_{\nu]\sigma})^+ \\
&= \mathbb{Q}H_{\mu\nu} - (\nabla^{\sigma}\Phi_{[\mu}\chi_{\nu]\sigma} + \nabla_{[\mu}\Phi^{\sigma}\chi_{\nu]\sigma})^- + (\Psi^{\sigma\rho}\Psi_{\rho[\mu}\chi_{\nu]\sigma})^- \\
&\qquad + (\Psi^{\sigma}{}_{[\mu}(H_{\nu]\sigma} - \mathsf{B}^-_{\nu]\sigma}))^- - (\Psi^{\sigma}{}_{[\mu}\mathsf{B}^+_{\nu]\sigma})^- + (\Psi^{\sigma}{}_{[\mu}\mathsf{B}^-_{\nu]\sigma})^+, \\
&= \mathbb{Q}H_{\mu\nu} + ((\nabla_{\mu}\Phi^{\sigma})\chi_{\sigma\nu} + (\nabla_{\nu}\Phi^{\sigma})\chi_{\mu\sigma})^- + (\Psi^{\sigma}{}_{[\mu}H_{\nu]\sigma})^- - (\Psi^{\sigma}{}_{[\mu}(\Psi^{\rho}{}_{\sigma}\chi_{\nu]\rho}))^- \\
&\qquad - (\Psi^{\sigma}{}_{[\mu}\mathsf{B}^-_{\nu]\sigma})^- - (\Psi^{\sigma}{}_{[\mu}\mathsf{B}^+_{\nu]\sigma})^- + (\Psi^{\sigma}{}_{[\mu}\mathsf{B}^-_{\nu]\sigma})^+.
\end{aligned} \tag{1.107}$$

Since Ψ is symmetric in its indices, we have

$$\Psi^{\sigma}{}_{[\mu}\mathsf{B}_{\nu]\sigma} = -\frac{1}{2}\Psi^{\sigma}{}_{[\mu}(\Psi^{\rho}{}_{\sigma}\chi_{\nu]\rho}), \tag{1.108}$$

and

$$(\Psi^{\sigma}{}_{[\mu}\mathsf{B}_{\nu]\sigma})^- = 0. \tag{1.109}$$

Hence, we find

$$\mathbb{Q}^2\chi_{\mu\nu} = \mathbb{Q}H_{\mu\nu} + ((\nabla_{\mu}\Phi^{\sigma})\chi_{\sigma\nu} + (\nabla_{\nu}\Phi^{\sigma})\chi_{\mu\sigma})^- + (\Psi^{\sigma}{}_{[\mu}H_{\nu]\sigma})^- + (\Psi^{\sigma}{}_{[\mu}\mathsf{B}^-_{\nu]\sigma})^+. \tag{1.110}$$

Thus, the only consistent choice to maintain (1.100) is that

$$\begin{aligned}
\mathbb{Q}H_{\mu\nu} = [\phi, \chi_{\mu\nu}] + \Phi^{\sigma}D_{\sigma}\chi_{\mu\nu} + ((\nabla_{\mu}\Phi^{\sigma})\chi_{\sigma\nu} + (\nabla_{\nu}\Phi^{\sigma})\chi_{\mu\sigma})^+ \\
- (\Psi^{\sigma}{}_{[\mu}H_{\nu]\sigma})^- + (\Psi^{\sigma}{}_{[\mu}(\Psi^{\rho}{}_{[\sigma}\chi_{\nu]]\rho})^-)^+,
\end{aligned} \tag{1.111}$$

which leads to

$$\begin{aligned}\mathbb{Q}^2\chi_{\mu\nu} &= [\phi,\chi_{\mu\nu}] + \Phi^\sigma D_\sigma\chi_{\mu\nu} + (\nabla_\mu\Phi^\sigma)\chi_{\sigma\nu} + (\nabla_\nu\Phi^\sigma)\chi_{\mu\sigma} \\ &= (\delta_\phi + \mathcal{L}^{(A)}_\Phi)\chi_{\mu\nu}. \end{aligned} \tag{1.112}$$

It will turn out that (1.111) is not only a consistent choice, but the correct choice, as we do have

$$\mathbb{Q}^2 H_{\mu\nu} = (\delta_\phi + \mathcal{L}^{(A)}_\Phi)H_{\mu\nu}. \tag{1.113}$$

We postpone the proof until the next section when we consider the bidegree splitting of $\mathbb{Q}$.

1.4.4 Summary

Our full Cartan model with anti-ghost modules is given by

$$\mathbb{Q}g_{\mu\nu} = \Psi_{\mu\nu}, \qquad \mathbb{Q}A_\mu = \psi_\mu, \tag{1.114}$$

$$\mathbb{Q}\Psi_{\mu\nu} = \nabla_\mu\Phi_\nu + \nabla_\nu\Phi_\mu, \qquad \mathbb{Q}\psi_\mu = -D_\mu\phi + \Phi^\sigma F_{\sigma\mu} \tag{1.115}$$

$$\mathbb{Q}\Phi^\sigma = 0, \qquad \mathbb{Q}\phi = -\Phi^\sigma\psi_\sigma, \tag{1.116}$$

$$\mathbb{Q}\lambda = \eta, \tag{1.117}$$

$$\mathbb{Q}\eta = [\phi,\lambda] + \Phi^\sigma D_\sigma\lambda, \tag{1.118}$$

$$\mathbb{Q}\chi_{\mu\nu} = H_{\mu\nu} - (\Psi^\sigma{}_{[\mu}\chi_{\nu]\sigma})^-, \tag{1.119}$$

$$\begin{aligned}\mathbb{Q}H_{\mu\nu} &= [\phi,\chi_{\mu\nu}] + \Phi^\sigma D_\sigma\chi_{\mu\nu} + ((\nabla_\mu\Phi^\sigma)\chi_{\sigma\nu} + (\nabla_\nu\Phi^\sigma)\chi_{\mu\sigma})^+ \\ &\quad - (\Psi^\sigma{}_{[\mu}H_{\nu]\sigma})^- + (\Psi^\sigma{}_{[\mu}(\Psi^\rho{}_{[\sigma}\chi_{\nu]]\rho})^-)^+. \end{aligned} \tag{1.120}$$

There are a few comments which can be made about this algebra. First, in terms of the the other self-dual auxiliary field D, which is related to H via

$$H_{\mu\nu} = F^{+}_{\mu\nu} - D_{\mu\nu}, \tag{1.121}$$

we have

$$\mathbb{Q}\chi_{\mu\nu} = F^{+}_{\mu\nu} - D_{\mu\nu} - (\Psi^{\sigma}{}_{[\mu}\chi_{\nu]\sigma})^{-}, \tag{1.122}$$

$$\begin{aligned} \mathbb{Q}D_{\mu\nu} = {} & 2(D_{[\mu}\psi_{\nu]})^{+} - [\phi, \chi_{\mu\nu}] - \Phi^{\sigma} D_{\sigma}\chi_{\mu\nu} - ((\nabla_{\mu}\Phi^{\sigma})\chi_{\sigma\nu} - (\nabla_{\nu}\Phi^{\sigma})\chi_{\sigma\mu})^{+} \\ & + (\Psi^{\sigma}{}_{[\mu}F^{-}_{\nu]\sigma})^{+} - (\Psi^{\sigma}{}_{[\mu}D_{\nu]\sigma})^{-} + \frac{1}{2}\Psi^{\rho\sigma}\Psi_{\rho[\mu}\chi_{\nu]\sigma}. \end{aligned} \tag{1.123}$$

We will typically use D when we are working with our action.

Turning to the closure of the algebra, let us begin by noting that both $H_{\mathcal{G}}(\mathcal{A}(P))$ and $H_{\mathsf{Diff}_{+}(\mathbb{X})}(\mathbb{X})$ sit as subalgebras of $H_{\mathbb{G}}(\mathbb{X})$. Each can be obtained by isolating to either side of the semi-direct product $\mathcal{G} \rtimes \mathsf{Diff}_{+}(\mathbb{X})$. Explicitly on the fields, turning off all gravity fields, expect the metric, we see that

$$\mathbb{Q}\big|_{\Psi=0,\Phi=0} = \mathcal{Q}. \tag{1.124}$$

Likewise, ignoring all gauge fields, we have

$$\mathbb{Q}\big|_{g,\Psi,\Phi} = \mathsf{d}. \tag{1.125}$$

This can effectively be summarized by splitting $\mathbb{Q}$ into bidegrees of (p, q) where p is the gauge degree inherited from the $H_{\mathcal{G}}(\mathcal{A})$ model shown in Table 3 and q is the gravity degree from the $H_{\mathsf{Diff}_+(\mathbb{X})}(\mathsf{Met}(\mathbb{X}))$ model shown in Table 4. We can then write

$$\mathbb{Q} = \mathbb{Q}^{(1,0)} + \mathbb{Q}^{(0,1)} + \mathbb{Q}^{(-1,2)}. \tag{1.126}$$

We denote the bidegree differentials as

$$\mathbb{Q}^{(1,0)} = \mathcal{Q}, \tag{1.127}$$

$$\mathbb{Q}^{(0,1)} = \widetilde{\mathsf{d}}, \tag{1.128}$$

$$\mathbb{Q}^{(-1,2)} = \mathsf{K} + \Delta_H. \tag{1.129}$$

Here $\mathcal{Q}$ acts as (1.4)-(1.7) and zero everywhere else. Next, $\widetilde{\mathsf{d}}$, as explained in Appendix D, is the lift of the differential d to the total space of the bundle $\Omega^{2,+}(\mathbb{X}, \mathsf{ad}\, P)$ over $\mathsf{Met}(\mathbb{X})$. It acts as d on g and Ψ as in (1.10) and (1.11) and, on our self-dual fields χ and H, it is the induced projected connection acting as

$$\widetilde{\mathsf{d}}\chi_{\mu\nu} = -(\Psi^{\sigma}{}_{[\mu}\chi_{\nu]\sigma})^{-}, \qquad \widetilde{\mathsf{d}}H_{\mu\nu} = -(\Psi^{\sigma}{}_{[\mu}H_{\nu]\sigma})^{-}. \tag{1.130}$$

On all other fields, it acts as zero. Finally, in a break from the other degrees we have $\mathsf{K}+\Delta_H$ in degree $(-1, 2)$. This splitting is given below in the nonzero transformations as

$$\mathsf{K}\psi_\mu = \Phi^\sigma F_{\sigma\mu}\ , \tag{1.131}$$

$$\mathsf{K}\phi = -\Phi^\sigma \psi_\sigma\ , \tag{1.132}$$

$$\mathsf{K}\eta = \Phi^\sigma D_\sigma \lambda\ , \tag{1.133}$$

$$\mathsf{K}H_{\mu\nu} = \Phi^\sigma D_\sigma \chi_{\mu\nu} + ((\nabla_\mu \Phi^\sigma)\chi_{\sigma\nu} - (\nabla_\nu \Phi^\sigma)\chi_{\sigma\mu})^+\ , \tag{1.134}$$

and

$$\Delta_H H_{\mu\nu} = (\Psi^{\sigma}{}_{[\mu}(\Psi^{\rho}{}_{[\sigma}\chi_{\nu]]\rho})^{-})^{+} \ . \tag{1.135}$$

We can, on all fields except (χ, H) (hence ignoring $\widetilde{\mathsf{d}}\big|_{\chi,H}$ and Δ_H), summarize the algebra with the relations

$$\mathcal{Q}^2 = \delta_\phi, \qquad \widetilde{\mathsf{d}}^2 = \mathcal{L}_\Phi\big|_{g,\Psi,\Phi}, \tag{1.136}$$

$$\mathsf{K}^2 = \{\mathcal{Q}, \widetilde{\mathsf{d}}\} = \{\widetilde{\mathsf{d}}, \mathsf{K}\} = 0, \qquad \{\mathcal{Q}, \mathsf{K}\} = \mathcal{L}_\Phi^{(A)}\big|_{F_A,\psi,\phi,\eta,\lambda}, \tag{1.137}$$

where we note that the final relation is only true on fields that transform in the adjoint representation (thus F_A as opposed to A).

The story is far more subtle on the self-dual fields, χ and H. There, we still have

$$\mathcal{Q}^2 = \delta_\phi, \qquad \mathsf{K}^2 = \{\mathcal{Q}, \widetilde{\mathsf{d}}\} = 0, \tag{1.138}$$

but now, on account of the metric dependence of the self-duality, the other relations change. First, we see that, due to Δ_H, there are new relations of

$$\{\mathcal{Q}, \Delta_H\}\chi_{\mu\nu} = (\Psi^{\sigma}{}_{[\mu}(\Psi^{\rho}{}_{[\sigma}\chi_{\nu]]\rho})^{-})^{+}, \qquad \Delta_H^2\chi_{\mu\nu} = \{\Delta_H, \mathsf{K}\}\chi_{\mu\nu} = 0, \tag{1.139}$$

$$\{\mathcal{Q}, \Delta_H\}H_{\mu\nu} = (\Psi^{\sigma}{}_{[\mu}(\Psi^{\rho}{}_{[\sigma}H_{\nu]]\rho})^{-})^{+}, \qquad \Delta_H^2 H_{\mu\nu} = \{\Delta_H, \mathsf{K}\}H_{\mu\nu} = 0. \tag{1.140}$$

Further, we have

$$\{\mathcal{Q}, \mathsf{K}\}\chi_{\mu\nu} = (\mathcal{L}_\Phi^{(A)}\chi_{\mu\nu})^{+}, \tag{1.141}$$

$$\{\mathcal{Q}, \mathsf{K}\}H_{\mu\nu} = (\mathcal{L}_\Phi^{(A)}H_{\mu\nu})^{+}, \tag{1.142}$$

exposing the failure to vary the Hodge star inside the self-dual projection. This is of course cured by the variation induced by the projected connection $\widetilde{\mathsf{d}}$ differential. Unfortunately, the cure comes with a symptom, as we have

$$\begin{aligned}\widetilde{\mathsf{d}}^2\chi_{\mu\nu} &= \frac{1}{2}\sqrt{g}\epsilon_{\mu\nu\alpha\beta}\left(\frac{1}{2}(\nabla_\sigma\Phi^\sigma)g^{\alpha\alpha'}g^{\beta\beta'} - (\nabla^\alpha\Phi^{\alpha'} + \nabla^{\alpha'}\Phi^\alpha)g^{\beta\beta'}\right)\chi_{\alpha'\beta'} + \frac{1}{2}\Psi^{\rho\sigma}\Psi_{\rho[\mu}\chi_{\nu]\sigma}, \\ &= (\mathcal{L}^{(A)}_\Phi\chi)_{\mu\nu} - (\mathcal{L}^{(A)}_\Phi\chi)^+_{\mu\nu} + \frac{1}{2}\Psi^{\rho\sigma}\Psi_{\rho[\mu}\chi_{\nu]\sigma} \qquad (1.143)\\ \widetilde{\mathsf{d}}^2 H_{\mu\nu} &= \frac{1}{2}\sqrt{g}\epsilon_{\mu\nu\alpha\beta}\left(\frac{1}{2}(\nabla_\sigma\Phi^\sigma)g^{\alpha\alpha'}g^{\beta\beta'} - (\nabla^\alpha\Phi^{\alpha'} + \nabla^{\alpha'}\Phi^\alpha)g^{\beta\beta'}\right)H_{\alpha'\beta'} + \frac{1}{2}\Psi^{\rho\sigma}\Psi_{\rho[\mu}H_{\nu]\sigma} \\ &= (\mathcal{L}^{(A)}_\Phi H)_{\mu\nu} - (\mathcal{L}^{(A)}_\Phi H)^+_{\mu\nu} + \frac{1}{2}\Psi^{\rho\sigma}\Psi_{\rho[\mu}H_{\nu]\sigma}. \qquad (1.144)\end{aligned}$$

As we recall from our discussion of the variation of self-dual forms, the first two terms in each computation are the variations of the Hodge star operator, and, when combined with (1.141)+(1.142) is precisely what is necessary to change the self-dual part of a Lie derivative to the normal Lie derivative. The addition terms are the curvature of the projected connection which can be rewritten as

$$\frac{1}{2}\Psi^{\rho\sigma}\Psi_{\rho[\mu}\chi_{\nu]\sigma} = -(\Psi^\sigma{}_{[\mu}(\Psi^\rho{}_{[\sigma}\chi_{\nu]]\rho})^-)^+, \qquad (1.145)$$

and

$$\frac{1}{2}\Psi^{\rho\sigma}\Psi_{\rho[\mu}H_{\nu]\sigma} = -(\Psi^\sigma{}_{[\mu}(\Psi^\rho{}_{[\sigma}H_{\nu]]\rho})^-)^+, \qquad (1.146)$$

as shown in Appendix E.1. Hence the potential problem terms are precisely the opposite of those in (1.139) and (1.140) respectively. This is good, but there is still need to check that the remaining anticommutators do not contribute any additional terms. On χ, this is simple, as we trivially find

$$\{\widetilde{\mathsf{d}}, \mathsf{K}\}\chi_{\mu\nu} = \{\widetilde{\mathsf{d}}, \Delta_H\}\chi_{\mu\nu} = 0. \qquad (1.147)$$

The story is not so simple on H. There we have

$$\{\widetilde{\mathsf{d}},\mathsf{K}\}H_{\mu\nu} = \frac{1}{2}\Psi_{\sigma[\mu}(\nabla^\rho\Phi^\sigma + \nabla^\sigma\Phi^\rho)\chi_{\nu]\rho} - \frac{1}{2}\Psi^{\sigma\rho}(\nabla_{[\mu}\Phi_\rho + \nabla_\rho\Phi_{[\mu})\chi_{\nu]\sigma}, \quad (1.148)$$

$$\{\widetilde{\mathsf{d}},\Delta_H\}H_{\mu\nu} = -\frac{1}{2}\Psi_{\sigma[\mu}(\nabla^\rho\Phi^\sigma + \nabla^\sigma\Phi^\rho)\chi_{\nu]\rho} + \frac{1}{2}\Psi^{\sigma\rho}(\nabla_{[\mu}\Phi_\rho + \nabla_\rho\Phi_{[\mu})\chi_{\nu]\sigma}, \quad (1.149)$$

so that

$$\{\widetilde{\mathsf{d}},\mathsf{K}\} = -\{\widetilde{\mathsf{d}},\Delta_H\}. \quad (1.150)$$

We squirrel the proof of this rather nontrivial relation away in Appendix E.2. All together, the relations (1.138)-(1.144) and (1.148)+(1.149) ensures that the localization multiplet can be consistently included in the model of diffeomorphism and gauge equivariant cohomology.

All together, we have shown that our transformation laws (1.114)-(1.120) satisfy the desired relation

$$\mathbb{Q}^2 = \delta_\phi + \mathcal{L}^{(A)}_\Phi \quad (1.151)$$

on all the fields of $H_{\mathbb{G}}(\mathbb{X})$ and its anti-ghost multiplets. We recognize this as a nontrivial result beyond the construct of the base of the combined Cartan model.

Before moving on to the next section, let us remark on the generality of our Cartan model. Outside of the self-dual fields, we have at no point made use of the dimension of $\mathbb{X}$. Supposing we had a smooth six manifold $\mathbb{Y}$, the self-dual fields would then be elements of $\Omega^{3,+}_g(\mathbb{Y},\mathsf{ad}\,P)$, so that χ and H are three form fields. Taking $\Delta_H H$ equal to negative the curvature of the projected connection on $\Omega^{3,+}_g(\mathbb{Y},\mathsf{ad}\,P)$ over $\mathsf{Met}(\mathbb{Y})$ and $\mathsf{K}H = (\mathcal{L}_\Phi\chi)^+$, we expect the algebra to once again close. The one nontrivial check would be to confirm that the relation $\{\widetilde{\mathsf{d}},\mathsf{K}\} = -\{\widetilde{\mathsf{d}},\Delta_H\}$ still holds on H. At present we do not have a geometric understanding of this relation in four dimensions, but it is reminiscent of some sort of Bianchi identity. This procedure could be also checked for any manifold of even dimension.

Returning to four dimensions, let us turn to the physical derivation of the model.

1.5 *Excursus*: Twisted Supergravity

Soon after Witten exposed the world to his topological twist, Karlhede and Roček identified the resulting transformation laws as a truncation and twist of $\mathcal{N} = 2$ conformal supergravity [47]. Likewise, but with far more complexity, the Cartan model with the anti-ghost multiplets just constructed is remarkably found lurking inside an appropriately truncated and twisted model of $\mathcal{N} = 2$ supergravity coupled to a vector multiplet on a symmetric gravitino background. In this section we will provide an criminally cursory review of Euclidean supergravity as constructed from superconformal gravity and then summarize the forthcoming work with Moore, Roček, Saxena, and the present author in [12] to arrive at the truncated and twisted theory.[34] The structure of this maneuver is summarized in Figure 2 below. In moving from (1) to (2) of the figure, we break the superconformal group by gauge fixing certain *compensating* fields multiplets as well as conforming to the so-call conventional constraints. (2) to (3) is a new to this work and full details are found in [12]. (3) to (4) is renormalization group flow, held as an exact process due to our $\mathbb{Q}$ supersymmetry. We will focus on the move from (3) to (4), and present the original work of specializing to a symmetric gravitino background which, after a field redefintion, is identical to the Cartan model of $H_{\mathbb{G}}(\mathbb{X})$.

We must disclaim that it is not our goal to provide any semblance of either introduction or review of supergravity, so we point the interested reader to the standard textbooks on the subject [31, 84]. Nevertheless, we will provide some light verbiage

[34] We note that our "twisted supergravity" is distinct from that of [11] for numerous reasons, the most salient of is that we are working with non-dynamical supergravity fields and our actions do not contain the Einstein-Hilbert term.

on the matter.

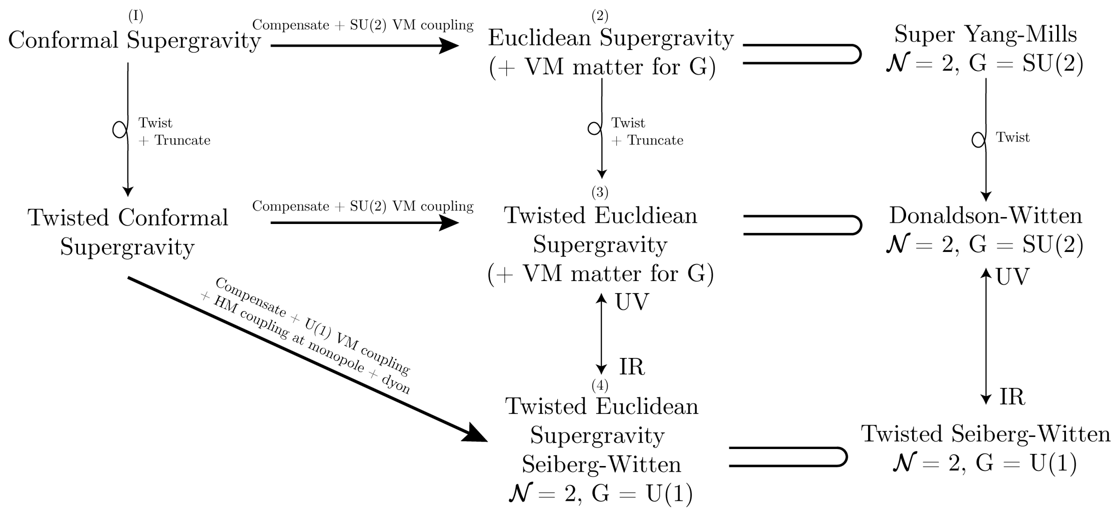

Figure 2: The above diagram summarizes the relationship between the various theories. We follow the number path.

1.5.1 Euclidean Supergravity

Supergravity is the gauge theory of the appropriate supersymmetry group of the symemtries of spacetime. Therefore, the supergravity of interest to us, namely Euclidean supergravity, is the gauge theory of the $\mathcal{N} = 2$ super Euclidean group (whose algebra is given by $\mathsf{SE}_{\mathcal{N}=2}$ in (0.50)+(0.51)). What this means is that for every symmetry, including the odd ones, we introduce both a connection and a local (dependent on spatial coordinates) transformation parameter. For a dynamical theory, all these fields are given kinetic terms and integrated over in the full theory, but for our purposes we will only consider these fields to be non-dynamical background entities.

To arrive at an off-shell formulation of these theories one starts with the superconformal group as opposed to the super Euclidean group.[35] The even part of this group is the conformal group $SO(5, 1)$, which contains transformations that preserve

[35]We point to [27] for a modern introduction to the representation theory of the superconformal groups.

angles. It contains the Euclidean group $\mathbb{R}^4 \rtimes \mathrm{SO}_0(4)$, the group of dilatations $\mathrm{SO}(1,1)$ generated by D, as well as the special conformal transformations K_μ.[36] In addition, we have R-symmetries, whose Lie algebra is $\mathfrak{su}(2)_\mathrm{R} \oplus \mathfrak{so}(1,1)_\mathrm{R}$. On the odd side of the superconformal algebra, we have the usual eight Q supersymmetries as well as the eight conformal supersymmetries S, which both can be split into left and right Weyl spinors.[37] Following [12], we collect this data together in the table below

Symmetry	Connection		Parameter
Translations	Vielbein	${e_\mu}^a$	$\boldsymbol{\xi}^a$
Rotations	Spin Connection	${\omega_\mu}^{ab}$	$\boldsymbol{\lambda}^{ab}$
Dilatations	Dilatation Connection	b_μ	$\boldsymbol{\Theta}^{(D)}$
Special Conformal Transformations	Special Conformal Connection	${f_\mu}^a$	${\boldsymbol{\Lambda}_K}^a$
$\mathfrak{su}(2)$ R Symmetry	$\mathfrak{su}(2)_\mathrm{R}$ Connection	${\omega_{\mathrm{R}\mu}}^i{}_j$	${\boldsymbol{\Theta}^i}_j$
$\mathfrak{su}(1,1)$ R Symmetry	$\mathfrak{su}(1,1)_\mathrm{R}$ Connection	$A_\mu^{(\mathrm{R})}$	${\boldsymbol{\Theta}^i}_j$
Supersymmetry	Gravitino	${\Psi_\mu}^{iA}, {\Psi_\mu}^{i\dot{A}}$	${\boldsymbol{\epsilon}_i}^A, {\overline{\boldsymbol{\epsilon}}_i}^{\dot{A}}$
S Supersymmetry	S Gravitino	${\mathrm{S}_\mu}^{iA}, {\mathrm{S}_\mu}^{i\dot{A}}$	${\boldsymbol{\eta}_i}^A, {\overline{\boldsymbol{\eta}}_i}^{\dot{A}}$

Table 5: The symmetry content of Euclidean $\mathcal{N}=2$ superconformal gravity.

Above, a, b are frame indices and all others are as in our original $\mathcal{N}=2$ supersymmetry of Section 0.2. We note that here, unlike in the case of regular $\mathcal{N}=2$ supersymmetry, each of the parameters is a function on $\mathbb{X}$, and thus, for example, $\partial_\mu {\boldsymbol{\epsilon}_i}^A \neq 0$. We also note that the metric of $\mathbb{X}$ is given by $g_{\mu\nu} = \eta_{ab} {e_\mu}^a {e_\nu}^b$, where $\eta_{ab} = \mathrm{diag}[1,1,1,1]$ is the flat Euclidean metric.

In addition to the gauge connections, to work off-shell, we must add three auxiliary fields. They are a scalar $\mathscr{D}$, a two form T_{ab}, and a spinor Ξ^i. All together, these fields

[36] The action of dialations take $x^\mu \mapsto \lambda x^\mu$ for some $\lambda \in \mathbb{R} - \{0\}$ and the special conformal transformations is the series of an inversion, a translation, followed by a final inversion and can be loosely understood as translations of infinity. Explicitly, the special conformal transformation maps $x^\mu \mapsto \frac{x^\mu - b^\mu x^2}{1 - 2x_\mu b^\mu + b^2 x^2}$.

[37] Our intuitive understanding of the special conformal transformation as the translation at infinity is further enhanced by the fact that the anti-commutator of the conformal supersymmetries is proportional K_μ. This is a direction analogy to the fact that supersymmetry Q squares to a translation P_μ. In fact, S is the conjugation of Q by the inversion operator.

make up the *Weyl multiplet*, which is an off-shell representation of the conformal superalgebra.

At present, each of the connections in Table 5 is an independent field, but this leads to an issue with the action of spatial translations. Therefore, to arrive at a physical theory one must impose the so called "conventional constraints." In superspace formalism, these are constraints on the form of the supertorsion, as more thoroughly explained in [58, 59]. For our purposes, they are conditions on the supercurvatures of spatial translations, supersymmetry, and rotations that can be algebraically solved. The result is that the gauge fields ${\omega_\mu}^{ab}$, ${f_\mu}^a$, and ${\mathrm{S}_\mu}^{iA}, {\mathrm{S}_\mu}^{i\dot{A}}$ are no longer independent and become *composite fields.*

This concludes the depth to which we review Euclidean supergravity, and now we turn to the truncation and twist.[38]

1.5.2 The Twist

Recall from our discussion of the original twist, that we chose an isomorphism between the $\mathrm{SU}(2)_\mathrm{R}$ principal bundle P_R and the $\mathrm{SO}(3)$ principal bundle P^+. This led to the relation

$$
{\omega_{\mathrm{R}\mu}}^{ij}\delta^A_j = \omega^{+AB}_\mu \delta^i_B \tag{1.152}
$$

This was recognized by Karlhede and Roček as a statement in conformal supergravity which led to the construction of the scalar supercharge $\mathcal{Q}$, thus giving Witten's original twist a physically motivated derivation.

In [12] it is realized that one can generalize this idea to keep not only a scalar $\mathcal{Q}$ but also an unconstrained local vector supersymmetry $\overline{\boldsymbol{\epsilon}}_{A\dot{A}}$. The associated connection

[38] For more extensive treatment of the subject, we direct those interested in the following. $\mathcal{N}=1$ supergravity in four dimensions with Lorentzian signature was developed in [28, 44, 45, 46]. This was was extended to $\mathcal{N}=2$ in the series of papers [14, 15, 16, 18, 19, 20]

for this vector supersymmetry is the vector gravitino ${\Psi_\mu}^{A\dot{A}}$. We can reexpress the gravitino in four components as $\Psi_{\mu\nu} = e_{\nu,A\dot{A}}{\Psi_\mu}^{A\dot{A}}$, which we crucially note is *not* entirely symmetric at present.

Further, one can truncated the Weyl multiplet of this theory through a series of non-trivial constraints on the gauge fields to arrive at a minimal theory which contains a scalar supersymmtry and a gravitino. In particular, we can gauge fix the fields

$$b_\mu = 0, \qquad {\Psi_\mu}^{AB} = 0, \tag{1.153}$$

$$A^{(\mathrm{R})}_\mu = 0, \qquad \Xi^{AB} = 0. \tag{1.154}$$

To ensure that these constraints are consistent, one must also show that this gauge fixing is maintained under a supersymmetry transformation. This requires one to check that that

$$\delta b_\mu = 0, \qquad \delta{\Psi_\mu}^{AB} = 0, \tag{1.155}$$

$$\delta A^{(\mathrm{R})}_\mu = 0, \qquad \delta\Xi^{AB} = 0. \tag{1.156}$$

These conditions lead to a cascade of further constraints on the gauge fields, which we will not reproduce here and instead point to [12]. All together, when the dust is settled, everything is consistent and the resulting independent fields are the metric $g_{\mu\nu}$, the (not entirely symmetric) gravitino $\Psi_{\mu\nu}$, and the bosonic anti-self-dual auxiliary field $T^-_{\mu\nu}$. In addition, the remaining supersymmetry parameters are the constant scalar $\boldsymbol{\epsilon}$, a vector $\overline{\boldsymbol{\epsilon}}^\mu$, a constant scalar $\boldsymbol{\eta}_0$, and a self-dual two form $\boldsymbol{\eta}^+_{\mu\nu}$.

1.5.3 Twisted Weyl Multiplet

The twisted and truncated Weyl multiplet transformations are foudn to be

$$\delta g_{\mu\nu} = \boldsymbol{\epsilon}\Psi_{(\mu\nu)}, \tag{1.157}$$

$$\delta\Psi_{(\mu\nu)} = \nabla_\mu\overline{\boldsymbol{\epsilon}}_\nu + \nabla_\nu\overline{\boldsymbol{\epsilon}}_\mu + \boldsymbol{\eta}_0 g_{\mu\nu}, \tag{1.158}$$

$$\delta\Psi_{[\mu\nu]} = \nabla_\mu\overline{\boldsymbol{\epsilon}}_\nu - \nabla_\nu\overline{\boldsymbol{\epsilon}}_\mu - \frac{1}{2}\boldsymbol{\epsilon}\Psi_\mu{}^\rho\Psi_{\nu\rho} - \boldsymbol{\epsilon}T^-_{[\mu\nu]} - \boldsymbol{\eta}^+_{[\mu\nu]}, \tag{1.159}$$

$$\delta T^-_{[\mu\nu]} = -\boldsymbol{\epsilon}\Psi_{[\mu}{}^\rho T^{-\sigma}{}_{\nu]}g_{\rho\sigma} + \overline{\boldsymbol{\epsilon}}^\rho R(Q)^-_{\mu\nu,\rho}, \tag{1.160}$$

where

$$R(Q)^-_{\mu\nu,\rho} = 2\mathcal{J}^-_{\mu\nu,\rho} - \left(g_{\rho[\mu}\mathcal{J}^+_{\nu]\sigma,\delta}g^{\delta\sigma} - g_{\rho[\mu}\mathcal{J}^-_{\nu]\sigma,\delta}g^{\delta\sigma}\right)^- \tag{1.161}$$

$$\mathcal{J}_{\mu\nu,\rho} = \nabla_{[\mu}\Psi_{\nu]\rho}. \tag{1.162}$$

In the above, we split $\Psi_{\mu\nu}$ into the symmetric and antisymmetric parts in order to prevent confusion with the symmetric gravitino $\Psi_{\mu\nu}$ from our construction of the Cartan model. We further note that various rescalings have been taken to streamline our analysis.

The goal of this section will be to show, that, after introducing BRST ghosts for the remaining local supersymmetries, we can consistently restrict to a background where the antisymmetric part of the gravitino vanishes. Moreover, the resulting algebra is exactly that of $H_{\mathsf{Diff}_+(\mathbb{X})}(\mathsf{Met}(\mathbb{X}))$.

We begin by introducing bosonic ghosts for the vector supersymmetry and the S supersymmetries. To do so, we split each fermionic transformation parameter into

$$\overline{\boldsymbol{\epsilon}}^\mu = \boldsymbol{\epsilon}\Phi^\mu, \quad \boldsymbol{\eta}_0 = \boldsymbol{\epsilon}\eta_0, \quad \text{and} \quad \boldsymbol{\eta}^+_{[\mu\nu]} = \boldsymbol{\epsilon}\eta^+_{[\mu\nu]}. \tag{1.163}$$

We note that this is a restriction of the transformation laws, as we have taken the fermionic part of each parameter to be identical to that of the constant scalar $\boldsymbol{\epsilon}$. At present, Φ^μ, η_0 and $\eta^+_{[\mu\nu]}$ are all unconstrained ghost fields. With all the fermionic variational parameters aligned, we can introduce the differential operators d_t, which, acting on any field A, is defined as

$$\delta A = \boldsymbol{\epsilon} d_\mathrm{t} A. \tag{1.164}$$

Extracting the parameter $\boldsymbol{\epsilon}$ from the transformation laws, we then arrive at[39]

$$d_\mathrm{t} g_{\mu\nu} = \Psi_{(\mu\nu)}, \tag{1.165}$$

$$d_\mathrm{t} \Psi_{(\mu\nu)} = \nabla_\mu \Phi_\nu + \nabla_\nu \Phi_\mu + g_{\mu\nu}\eta_0, \tag{1.166}$$

$$d_\mathrm{t} \Psi_{[\mu\nu]} = \nabla_\mu \Phi_\nu - \nabla_\nu \Phi_\mu - \frac{1}{2}\Psi_\mu{}^\rho \Psi_{\nu\rho} - T^-_{[\mu\nu]} - \eta^+_{[\mu\nu]}, \tag{1.167}$$

$$d_\mathrm{t} T^-_{[\mu\nu]} = \Psi_{[\mu}{}^\rho T^{-\sigma}{}_{\nu]} g_{\rho\sigma} + \Phi^\rho R(Q)^-_{\mu\nu,\rho}. \tag{1.168}$$

Here, in order to agree with the Cartan model, we append this list with

$$d_\mathrm{t} \Phi^\mu = 0. \tag{1.169}$$

With "cosmetics" aside, we begin our approach to the Cartan model of equivariant cohomology in earnest. First, we freely set

$$\eta_0 = 0, \quad \text{and} \quad d_\mathrm{t}\eta_0 = 0. \tag{1.170}$$

[39]This procedure may cause confusion for those steeped in the formalities of SUGRA. For these esteemed colleagues, we present the following construction, which is functionally identical. We introducing a Grassmann valued constant $\boldsymbol{\Lambda}$ and a constant commuting scalar supersymmetry ghost c_ϵ. Then $\boldsymbol{\epsilon}$ is formally replaced by $\boldsymbol{\Lambda} c_\epsilon$, $\overline{\boldsymbol{\epsilon}}^\mu$ by $\boldsymbol{\Lambda}\Phi^\mu$, and $\boldsymbol{\eta}^+_{\mu\nu}$ by $\boldsymbol{\Lambda} c^+_{\mu\nu}$ for a self-dual commuting two form ghost $c^+_{\mu\nu}$. We then restrict to the subspace where $\boldsymbol{\eta}_0 = 0$ and choose $c_\epsilon = 1$. Finally, we define the differential d_t on a field A as $\delta A = \boldsymbol{\Lambda} d_\mathrm{t} A$.

Next to turn off the antisymmetric part of the gravitino and set $\Psi_{[\mu\nu]} = 0$, we must enforce $d_{\mathrm{t}}\Psi_{[\mu\nu]} = 0$, so that no antisymmetric parts sneak in through a supersymmetry transformation. Since $\eta^{+}_{[\mu\nu]}$ is unconstrained, the self-dual part of $d_{\mathrm{t}}\Psi_{[\mu\nu]} = 0$ can be set to zero by restricting to a background where

$$\eta^{+}_{[\mu\nu]} = (\nabla_\mu \Phi_\nu - \nabla_\nu \Phi_\mu)^{+} - \frac{1}{2}(\Psi_\mu{}^{\rho}\Psi_{\nu\rho})^{+}. \tag{1.171}$$

On the anti self-dual side, we have natural choice of

$$T^{-}_{[\mu\nu]} = (\nabla_\mu \Phi_\nu - \nabla_\nu \Phi_\mu)^{-} - \frac{1}{2}(\Psi_\mu{}^{\rho}\Psi_{\nu\rho})^{-}. \tag{1.172}$$

Here, even though $T^{-}_{[\mu\nu]}$ is an auxiliary field, it is not fully unconstrained and has a fixed variation in the Weyl multiplet. Thus, in order for this background value of $T^{-}_{[\mu\nu]}$ to be consistent with the theory, we also need to check that

$$\Psi_{[\mu}{}^{\rho}T^{-\sigma}{}_{\nu]}g_{\rho\sigma} + \Phi^{\rho}R(Q)^{-}_{\mu\nu,\rho} \stackrel{?}{=} d_{\mathrm{t}}\left((\nabla_\mu \Phi_\nu - \nabla_\nu \Phi_\mu)^{-} - \frac{1}{2}(\Psi_\mu{}^{\rho}\Psi_{\nu\rho})^{-}\right) \tag{1.173}$$

Beginning on the left hand side, let us interrogate the supercurvature $R(Q)$. Exploiting the full extent of the self-dual and anti-self-dual projections, we have

$$\begin{aligned} R(Q)^{-}_{\mu\nu,\rho} &= (\nabla_\mu \Psi_{\nu\rho} - \nabla_\nu \Psi_{\mu\rho})^{-} + \frac{1}{2}g^{\delta\sigma}\left(g_{\rho\mu}\left(\mathcal{J}^{-}_{\nu\sigma,\delta} - \mathcal{J}^{+}_{\nu\sigma,\delta}\right) - g_{\rho\nu}\left(\mathcal{J}^{-}_{\mu\sigma,\delta} - \mathcal{J}^{+}_{\mu\sigma,\delta}\right)\right) \\ &= (\nabla_\mu \Psi_{\nu\rho} - \nabla_\nu \Psi_{\mu\rho})^{-} - \frac{1}{4}\sqrt{g}g^{\delta\sigma}\left(g_{\rho\mu}\epsilon_{\nu\sigma\lambda\eta}\mathcal{J}^{\lambda\eta}{}_{\delta} - g_{\rho\nu}\epsilon_{\mu\sigma\lambda\eta}\mathcal{J}^{\lambda\eta}{}_{\delta}\right), \\ &= (\nabla_\mu \Psi_{\nu\rho} - \nabla_\nu \Psi_{\mu\rho})^{-} - \frac{1}{4}\sqrt{g}\left(g_{\rho\mu}\epsilon_{\nu\sigma\lambda\eta}\mathcal{J}^{\lambda\eta,\sigma} - g_{\rho\nu}\epsilon_{\mu\sigma\lambda\eta}\mathcal{J}^{\lambda\eta,\sigma}\right), \end{aligned} \tag{1.174}$$

where we have use the defintional fact that

$$\mathcal{J}^{\pm}_{\mu\nu,\rho} = \frac{1}{2}\mathcal{J}_{\mu\nu,\rho} \pm \frac{1}{4}\sqrt{g}\epsilon_{\mu\nu\lambda\eta}\mathcal{J}^{\lambda\eta}{}_{\rho}. \tag{1.175}$$

Next, given that we are working under the assumption that are gravitinos are now completely symmetric, it follows that $\mathcal{J}^{\mu\nu\sigma}$ will be a sum of two terms which each have two symmetric indices. Thus, when fully contracted with the completely anti-symmetric tensor as in the second term of $R(Q)^-_{\mu\nu,\rho}$ above, both terms vanishes. Therefore we have

$$R(Q)^-_{\mu\nu,\rho} = (\nabla_\mu \Psi_{\nu\rho} - \nabla_\nu \Psi_{\mu\rho})^-. \tag{1.176}$$

Pluggin this back into $d_t T^-_{[\mu\nu]}$ along with our choice of $T^-_{[\mu\nu]}$ in (1.172), we have the left hand side of (1.173) as

$$\begin{aligned} d_t T^-_{[\mu\nu]} &= -(\Psi^\sigma{}_{[\mu} T^-_{\nu]\sigma}) + \Phi^\sigma (\nabla_\mu \Psi_{\nu\sigma} - \nabla_\nu \Psi_{\mu\sigma})^- \\ &= -(\Psi^\sigma{}_{[\mu} T^-_{\nu]\sigma})^+ - (\Psi^\sigma{}_{[\mu} T^-_{\nu]\sigma})^- + \Phi^\sigma (\nabla_\mu \Psi_{\nu\sigma} - \nabla_\nu \Psi_{\mu\sigma})^- \end{aligned} \tag{1.177}$$

Turning to the right hand side of (1.173), keeping our lessons about varying anti-self-dual conditions in mind, we compute

$$\begin{aligned} d_t T^-_{[\mu\nu]} &= d_t \left((\nabla_\mu \Phi_\nu - \nabla_\nu \Phi_\mu)^- - \frac{1}{2} (\Psi_\mu{}^\rho \Psi_{\nu\rho})^- \right) \\ &= -(\Psi^\sigma{}_{[\mu} T^-_{\nu]\sigma})^+ + \left(\Psi^\sigma{}_{[\mu} \left((\nabla_{\nu]} \Phi_\sigma - \nabla_\sigma \Phi_{\nu]})^+ - \frac{1}{2} (\Psi_{\nu]}{}^\rho \Psi_{\sigma\rho})^+ \right) \right)^- \\ &\quad + d_t \left((\nabla_\mu \Phi_\nu - \nabla_\nu \Phi_\mu) - \frac{1}{2} (\Psi_\mu{}^\rho \Psi_{\nu\rho}) \right)^- \\ &= -(\Psi^\sigma{}_{[\mu} T^-_{\nu]\sigma})^+ + \left(\Psi^\sigma{}_{[\mu} \left((\nabla_{\nu]} \Phi_\sigma - \nabla_\sigma \Phi_{\nu]})^+ - \frac{1}{2} (\Psi_{\nu]}{}^\rho \Psi_{\sigma\rho})^+ \right) \right)^- \\ &\quad + \frac{1}{2} \Psi^{\rho\sigma} (\Psi_{\mu\sigma} \Psi_{\nu\rho})^- + \frac{1}{2} (\Psi_{\nu\sigma} \nabla_\mu \Phi^\sigma - \Psi_{\mu\sigma} \nabla_\nu \Phi^\sigma)^- \\ &\quad + \frac{1}{2} (\Psi_\mu{}^\sigma \nabla_\sigma \Phi_\nu - \Psi_\nu{}^\sigma \nabla_\sigma \Phi_\mu)^- + \Phi^\sigma \left(\nabla_\mu \Psi_{\nu\sigma} - \nabla_\nu \Psi_{\mu\sigma} \right)^- \\ &= -(\Psi^\sigma{}_{[\mu} T^-_{\nu]\sigma})^+ - \left(\Psi^\sigma{}_{[\mu} (\nabla_{\nu]} \Phi_\sigma - \nabla_\sigma \Phi_{\nu]})^- \right)^- + \Phi^\sigma \left(\nabla_\mu \Psi_{\nu\sigma} - \nabla_\nu \Psi_{\mu\sigma} \right)^- \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\left(\Psi^{\sigma}{}_{[\mu}(\Psi_{\nu]}{}^{\rho}\Psi_{\sigma\rho})^{+}\right)^{-}+\frac{1}{2}(\Psi_{\mu}{}^{\sigma}(\Psi_{\nu}{}^{\rho}\Psi_{\sigma\rho}))^{-} \\
=&-(\Psi^{\sigma}{}_{[\mu}T^{-}_{\nu]\sigma})^{+}-\left(\Psi^{\sigma}{}_{[\mu}\left((\nabla_{\nu]}\Phi_{\sigma}-\nabla_{\sigma}\Phi_{\nu]})^{-}-\frac{1}{2}(\Psi_{\nu]}{}^{\rho}\Psi_{\sigma\rho})^{-}\right)\right)^{-} \\
&+\Phi^{\sigma}\left(\nabla_{\mu}\Psi_{\nu\sigma}-\nabla_{\nu}\Psi_{\mu\sigma}\right)^{-} \\
=&-(\Psi^{\sigma}{}_{[\mu}T^{-}_{\nu]\sigma})^{+}-(\Psi^{\sigma}{}_{[\mu}T^{-}_{\nu]\sigma})^{-}+\Phi^{\sigma}(\nabla_{\mu}\Psi_{\nu\sigma}-\nabla_{\nu}\Psi_{\mu\sigma})^{-}
\end{aligned}
\tag{1.178}
$$

Joyfully, we can then strip (1.173) of the question mark looming over the equality, having verified that the choice

$$
T^{-}_{[\mu\nu]}=(\nabla_{\mu}\Phi_{\nu}-\nabla_{\nu}\Phi_{\mu})^{-}-\frac{1}{2}(\Psi_{\mu}{}^{\rho}\Psi_{\nu\rho})^{-}. \tag{1.179}
$$

is consistent.

In summary, we have started with the algebra of the Weyl mutiplet of tSUGRA as

$$
d_{\mathrm{t}}g_{\mu\nu}=\Psi_{(\mu\nu)}, \tag{1.180}
$$

$$
d_{\mathrm{t}}\Psi_{(\mu\nu)}=\nabla_{\mu}\Phi_{\nu}+\nabla_{\nu}\Phi_{\mu}+g_{\mu\nu}\eta_0 \tag{1.181}
$$

$$
d_{\mathrm{t}}\Psi_{[\mu\nu]}=\nabla_{\mu}\Phi_{\nu}-\nabla_{\nu}\Phi_{\mu}-\frac{1}{2}\Psi_{\mu}{}^{\rho}\Psi_{\nu\rho}-T^{-}_{[\mu\nu]}-\eta^{+}_{[\mu\nu]} \tag{1.182}
$$

$$
d_{\mathrm{t}}T^{-}_{[\mu\nu]}=\Psi_{[\mu}{}^{\rho}T^{-\sigma}{}_{\nu]}g_{\rho\sigma}+\Phi^{\rho}R(Q)^{-}_{\mu\nu,\rho} \tag{1.183}
$$

Then enforcing the following *consistent* constraints,

$$
d_{\mathrm{t}}\Phi^{\mu}=0,\qquad \Psi_{[\mu\nu]}=0,\qquad \eta_0=0, \tag{1.184}
$$

$$
\eta^{+}_{[\mu\nu]}=(\nabla_{\mu}\Phi_{\nu}-\nabla_{\nu}\Phi_{\mu})^{+}-\frac{1}{2}(\Psi_{\mu}{}^{\rho}\Psi_{\nu\rho})^{+}, \tag{1.185}
$$

$$
T^{-}_{[\mu\nu]}=(\nabla_{\mu}\Phi_{\nu}-\nabla_{\nu}\Phi_{\mu})^{-}-\frac{1}{2}(\Psi_{\mu}{}^{\rho}\Psi_{\nu\rho})^{,} \tag{1.186}
$$

when we restrict to the symmetric gravitino background, the nontrivial part of the algebra reduces to

$$d_{\mathrm{t}} g_{\mu\nu} = \Psi_{\mu\nu}, \tag{1.187}$$

$$d_{\mathrm{t}} \Psi_{\mu\nu} = \nabla_\mu \Phi_\nu + \nabla_\nu \Phi_\mu, \tag{1.188}$$

$$d_{\mathrm{t}} \Phi^\mu = 0. \tag{1.189}$$

This is precisely the transformation laws of $H_{\mathsf{Diff}_+(\mathbb{X})}(\mathsf{Met}(\mathbb{X}))$ in (1.10)+(1.11). Never satisfied, let us see how to incorporate the vector multiplet fields into the twisted conformal supergravity approach.

1.5.4 Twisted Vector Multiplet

It is a typical procedure to couple conformal supergravity to a $\mathcal{N} = 2$ vector multiplet, as can be found in [17, 31, 54]. Under the truncation and twisting, and before moving to the symmetric gravitino background, the fields transform as

$$\delta A_\mu = \boldsymbol{\epsilon}\psi_\mu + \overline{\boldsymbol{\epsilon}}^\sigma \chi_{\sigma\mu} - \overline{\boldsymbol{\epsilon}}_\mu \eta + \overline{\boldsymbol{\epsilon}}^\sigma \Psi_{\mu\sigma}\lambda, \tag{1.190}$$

$$\delta\psi_\mu = \frac{1}{2}\boldsymbol{\epsilon}\Psi_\mu{}^\sigma\psi_\sigma - \boldsymbol{\epsilon}\mathbf{D}_\mu\phi + \overline{\boldsymbol{\epsilon}}^\sigma(\widehat{\mathbf{F}}^-_{\sigma\mu} + \lambda T^-_{\sigma\mu} + D_{\sigma\mu}) - \overline{\boldsymbol{\epsilon}}_\mu[\lambda, \phi], \tag{1.191}$$

$$\delta\phi = -\overline{\boldsymbol{\epsilon}}^\sigma \psi_\sigma, \tag{1.192}$$

$$\delta\eta = \overline{\boldsymbol{\epsilon}}^\sigma \mathbf{D}_\sigma \lambda - \boldsymbol{\epsilon}[\lambda, \phi] - \boldsymbol{\eta}_0\lambda, \tag{1.193}$$

$$\delta\lambda = \boldsymbol{\epsilon}\eta, \tag{1.194}$$

$$\delta\chi_{\mu\nu} = -\boldsymbol{\epsilon}(\Psi_{[\mu}{}^\sigma\chi_{\nu]\sigma}) - 4(\overline{\boldsymbol{\epsilon}}_{[\mu}\mathbf{D}_{\nu]}\lambda)^+ + \boldsymbol{\epsilon}(\widehat{\mathbf{F}}^+_{\mu\nu} - D_{\mu\nu}) + \boldsymbol{\eta}^+_{\mu\nu}\lambda, \tag{1.195}$$

$$\begin{aligned}\delta D_{\mu\nu} = {} & \boldsymbol{\epsilon}(\Psi_{[\mu}{}^\sigma D_{\nu]\sigma}) + 2\boldsymbol{\epsilon}(\mathbf{D}_{[\mu}\psi_{\nu]})^+ + 2(\overline{\boldsymbol{\epsilon}}_{[\mu}\mathbf{D}^\sigma\chi_{\nu]\sigma})^+ - 2(\overline{\boldsymbol{\epsilon}}_{[\mu}\mathbf{D}_{\nu]}\eta)^+ \\ & - \boldsymbol{\epsilon}[\phi, \chi_{\mu\nu}] + 4(\overline{\boldsymbol{\epsilon}}_{[\mu}[\phi, \psi_{\nu]}])^+,\end{aligned} \tag{1.196}$$

which is expressed supercovariantly. The supercovariant expressions are given by

$$\mathbf{D}_\mu A_\nu = D_\mu A_\nu - \frac{1}{2}\Psi_\mu{}^\sigma \chi_{\sigma\nu} + \frac{1}{2}\Psi_{\mu\nu}\eta - \Psi_\mu{}^\sigma \Psi_{\nu\sigma}\lambda, \tag{1.197}$$

$$\mathbf{D}_\mu \psi_\nu = D_\mu \psi_\nu - \frac{1}{2}\Psi_\mu{}^\sigma(\widehat{\mathbf{F}}^-_{\sigma\nu} + \lambda T^-_{\sigma\nu} + D_{\sigma\nu}) + \frac{1}{2}\Psi_{\mu\nu}[\lambda, \phi], \tag{1.198}$$

$$\mathbf{D}_\mu \phi = D_\mu \phi + \frac{1}{2}\Psi_\mu{}^\sigma \psi_\sigma, \tag{1.199}$$

$$\mathbf{D}_\mu \lambda = D_\mu \lambda, \tag{1.200}$$

$$\mathbf{D}_\mu \eta = D_\mu \eta - \frac{1}{2}\Psi_\mu{}^\sigma D_\sigma \lambda + \frac{1}{2\sqrt{2}}\lambda \mathrm{S}_\mu, \tag{1.201}$$

$$\mathbf{D}_\mu \chi_{\nu\sigma} = D_\mu \chi_{\nu\sigma} + 2(\Psi_{\mu[\nu} D_{\sigma]}\lambda)^+ - \sqrt{2}\lambda \mathrm{S}^+_{\mu,[\nu\sigma]}, \tag{1.202}$$

$$\widehat{\mathbf{F}}_{\mu\nu} = F_{\mu\nu} + \Psi_{[\mu}{}^\sigma \chi_{\nu]\sigma} + \Psi_{[\mu\nu]}\eta + \frac{1}{2}\Psi_\mu{}^\sigma \Psi_{\nu\sigma}\lambda. \tag{1.203}$$

These supercovariant objects are built by requiring that the supersymmetric variations, say, computed $\delta(\mathbf{D}_{\boldsymbol{\mu}} A)$ for some field A, do not contain any derivatives of the supersymmetric variation parameters. We arrive at the supercovariant curvature in a similar fashion. Finally, the S_μ and $\mathrm{S}^+_{\mu,[\nu\sigma]}$ are connections for the S-supersymmetry, having split into the scalar and self-dual symmetries respectively under the twist. They are given by

$$\mathrm{S}_\mu = \frac{1}{\sqrt{2}}(-\nabla^\sigma \Psi_{\mu\sigma} + \nabla_\mu \Psi_\sigma{}^\sigma) \tag{1.204}$$

$$\mathrm{S}^+_{\mu,[\nu\sigma]} = \frac{1}{\sqrt{2}}(\nabla_\nu \Psi_{\sigma\mu} - \nabla_\sigma \Psi_{\nu\mu})^+ + \frac{1}{\sqrt{2}}\left(-g_{\mu[\nu}\nabla^\rho \Psi_{\sigma]\rho} + g_{\mu[\nu}\nabla_{\sigma]}\Psi_\rho{}^\rho\right)^+ \tag{1.205}$$

As with the twisted Weyl multiplet, we have taken various field rescalings in order to decrease the clutter in our computations. Next, let us reintroduce the bosonic ghosts of the previous section and restrict to the symmetric gravitino background. Writing the supercovariant terms explicitly, we then obtain

$$d_{\mathrm{t}}A_\mu = \psi_\mu + \Phi^\sigma\chi_{\sigma\mu} - \Phi_\mu\eta - \Phi^\sigma\Psi_{\mu\sigma}\lambda, \tag{1.206}$$

$$d_{\mathrm{t}}\psi_\mu = -D_\mu\phi + \Phi^\sigma\left(F^-_{\sigma\mu} + D_{\sigma\mu} + (\Psi_{[\sigma}{}^{\rho}\chi_{\mu]\rho})^- + 2\lambda(\nabla_{[\sigma}\Phi_{\mu]})^- - g_{\mu\sigma}[\lambda,\phi]\right), \tag{1.207}$$

$$d_{\mathrm{t}}\phi = -\Phi^\sigma\psi_\sigma, \tag{1.208}$$

$$d_{\mathrm{t}}\lambda = \eta, \tag{1.209}$$

$$d_{\mathrm{t}}\eta = \Phi^\sigma D_\sigma\lambda - [\lambda,\phi], \tag{1.210}$$

$$d_{\mathrm{t}}\chi_{\mu\nu} = (\Psi_{[\mu}{}^{\sigma}\chi_{\nu]\sigma})^- - 4(\Phi_{[\mu}D_{\nu]}\lambda)^+ + (F^+_{\mu\nu} - D_{\mu\nu}) + 2(\nabla_{[\mu}\Phi_{\nu]})^+\lambda \tag{1.211}$$

$$\begin{aligned} d_{\mathrm{t}}D_{\mu\nu} = &-(\Psi_{[\mu}{}^{\sigma}D_{\nu]\sigma})^- + 2(D_{[\mu}\psi_{\nu]})^+ + (\Psi_{[\mu}{}^{\sigma}F^-_{\nu]\sigma})^+ - [\phi,\chi_{\mu\nu}] + 4(\Phi_{[\mu}[\lambda,\psi_{\nu]}])^+ \\ &- \frac{1}{2}\left(\Psi_{[\mu}{}^{\sigma}(\Psi_{\nu]}{}^{\rho}\chi_{\sigma\rho} - \Psi_{\sigma}{}^{\rho}\chi_{\nu]\rho})^-\right)^+ + \lambda\left(\Psi_{[\mu}{}^{\sigma}(\nabla_{\nu]}\Phi_\sigma - \nabla_\sigma\Phi_{\nu]})^-\right)^+ \\ &- 2(\Phi_{[\mu}D_{\nu]}\eta)^+ + 2(\Phi_{[\mu}D^\sigma\chi_{\nu]\sigma})^+ + \frac{1}{\sqrt{2}}(\Phi_{[\mu}\lambda\mathrm{S}_{\nu]})^+ - 2\sqrt{2}(\Phi_{[\mu}\lambda\mathrm{S}^{+\sigma}{}_{\nu]\sigma})^+ \\ &+ (\Phi_{[\mu}(\Psi_{\nu]}{}^{\sigma}D_\sigma\lambda))^+ - 2\left(\Phi_{[\mu}(\Psi^\sigma{}_{\nu]}D_\sigma\lambda - \Psi^\sigma{}_\sigma D_{\nu]}\lambda)^+\right)^+. \end{aligned} \tag{1.212}$$

Here there are a few simplifications in $d_{\mathrm{t}}D_{\mu\nu}$ on account of now having an entirely symmetric gravitino. First, by writing the self-dual projections explicitly, we have

$$\begin{aligned} \mathrm{S}^{+\sigma}{}_{\nu\sigma} &= g^{\sigma\rho}\left(\frac{1}{\sqrt{2}}(\nabla_\nu\Psi_{\sigma\rho} - \nabla_\sigma\Psi_{\nu\rho})^+ + \frac{1}{\sqrt{2}}(-g_{\rho[\nu}\nabla^\lambda\Psi_{\sigma]\lambda} + g_{\rho[\nu}\nabla_{\sigma]}\Psi^\lambda{}_\lambda)^+\right) \\ &= \frac{1}{2\sqrt{2}}\left(-\nabla^\sigma\Psi_{\nu\sigma} + \nabla_\nu\Psi^\sigma{}_\sigma\right) + \frac{1}{4\sqrt{2}}\sqrt{g}\epsilon_{\nu\sigma\alpha\beta}g^{\sigma\rho}g^{\alpha\alpha'}g^{\beta\beta'}(\nabla_{\alpha'}\Psi_{\beta'\rho} - \nabla_{\beta'}\Psi_{\alpha'\rho}) \\ &\quad + \frac{1}{4\sqrt{2}}\sqrt{g}\epsilon_{\nu\sigma\alpha\beta}g^{\sigma\rho}g^{\alpha\alpha'}g^{\beta\beta'}(-g_{\rho[\alpha'}\nabla^\lambda\Psi_{\beta']\lambda} + g_{\rho[\alpha'}\nabla_{\beta']}\Psi^\lambda{}_\lambda) \\ &\quad + \frac{1}{2\sqrt{2}}g^{\sigma\rho}(-g_{\rho[\nu}\nabla^\lambda\Psi_{\sigma]\lambda} + g_{\rho[\nu}\nabla_{\sigma]}\Psi^\lambda{}_\lambda). \end{aligned} \tag{1.213}$$

In the above, each expression with the epsilon tensor vanishes under considerations of symmetric indices. Thus, we continue to

$$\begin{aligned}
\mathrm{S}^{+\sigma}{}_{\nu\sigma} &= \frac{1}{2\sqrt{2}}\left(-\nabla^\sigma\Psi_{\nu\sigma} + \nabla_\nu\Psi^\sigma{}_\sigma\right) + \frac{1}{2\sqrt{2}} g^{\sigma\rho}(-g_{\rho[\nu}\nabla^\lambda\Psi_{\sigma]\lambda} + g_{\rho[\nu}\nabla_{\sigma]}\Psi^\lambda{}_\lambda) \\
&= \left(\frac{1}{2\sqrt{2}} + \frac{1}{4\sqrt{2}}\right)\left(-\nabla^\sigma\Psi_{\nu\sigma} + \nabla_\nu\Psi^\sigma{}_\sigma\right) - \frac{1}{4\sqrt{2}}(-\delta^\sigma{}_\sigma\nabla^\lambda\Psi_{\nu\lambda} + \delta^\sigma{}_\sigma\nabla_\nu\Psi^\lambda{}_\lambda) \\
&= -\frac{1}{4\sqrt{2}}\left(-\nabla^\sigma\Psi_{\nu\sigma} + \nabla_\nu\Psi^\sigma{}_\sigma\right) = -\frac{1}{4}\mathrm{S}_\nu.
\end{aligned} \tag{1.214}$$

Noting the analogous index structure, an identical argument leads us to see that

$$\left(\Phi_{[\mu}(\Psi^\sigma{}_{\nu]}D_\sigma\lambda - \Psi^\sigma{}_\sigma D_{\nu]}\lambda)^+\right)^+ = \frac{1}{2}\left(\Phi_{[\mu}(\Psi^\sigma{}_{\nu]}D_\sigma\lambda - \Psi^\sigma{}_\sigma D_{\nu]}\lambda)\right)^+. \tag{1.215}$$

Using the above identity to rewrite the last term in (1.212), we can collect all terms of the form $\sim \Phi\Psi D_A\lambda$ together to compute

$$\begin{aligned}
(d_{\mathrm{t}}D_{\mu\nu})_{\Phi\Psi D_A\lambda} &= (\Phi_{[\mu}(\Psi_{\nu]}{}^\sigma D_\sigma\lambda))^+ - \left(\Phi_{[\mu}(\Psi^\sigma{}_{\nu]}D_\sigma\lambda - \Psi^\sigma{}_\sigma D_{\nu]}\lambda)\right)^+ \\
&= -2(\Psi^\sigma{}_{[\mu}(\Phi_{\nu]}D_\sigma\lambda))^+ - \Psi^\sigma{}_\sigma(\Phi_{[\mu}D_{\nu]}\lambda)^+ \\
&= -2(\Psi^\sigma{}_{[\mu}(\Phi_{\nu]}D_\sigma\lambda))^+ + 2(\Psi^\sigma{}_{[\mu}(\Phi_{\nu]}D_\sigma\lambda - \Phi_\sigma D_{\nu]}\lambda)^+)^+ \\
&= -2(\Psi^\sigma{}_{[\mu}(\Phi_{\nu]}D_\sigma\lambda))^+ + 2(\Psi^\sigma{}_{[\mu}(\Phi_{\nu]}D_\sigma\lambda - \Phi_\sigma D_{\nu]}\lambda))^+ \\
&\qquad - 2(\Psi^\sigma{}_{[\mu}(\Phi_{\nu]}D_\sigma\lambda - \Phi_\sigma D_{\nu]}\lambda)^-)^+ \\
&= -2(\Psi^\sigma{}_{[\mu}(\Phi_\sigma D_{\nu]}\lambda))^+ - 2(\Psi^\sigma{}_{[\mu}(\Phi_{\nu]}D_\sigma\lambda - \Phi_\sigma D_{\nu]}\lambda)^-)^+.
\end{aligned} \tag{1.216}$$

These simplifications lead us to the final form of

$$\begin{aligned}
d_{\mathrm{t}}D_{\mu\nu} = {} & 2(D_{[\mu}\psi_{\nu]})^+ - [\phi, \chi_{\mu\nu}] - (\Psi^\sigma{}_{[\mu}D_{\nu]\sigma})^- + 2(\Phi_{[\mu}D^\sigma\chi_{\nu]\sigma})^+ - 2(\Phi_{[\mu}D_{\nu]}\eta)^+ \\
& + \left(\Psi^\sigma{}_{[\mu}\left(F^-_{\nu]\sigma} + \lambda(\nabla_{\nu]}\Phi_\sigma - \nabla_\sigma\Phi_{\nu]})^- - (\Psi^\rho{}_{[\sigma}\chi_{\nu]]\rho})^-\right)\right)^+ - 4(\Phi_{[\mu}[\psi_{\nu]}, \lambda])^+ \\
& - 2\Phi^\sigma(\Psi_{\sigma[\mu}D_{\nu]}\lambda)^+ - 2\left(\Psi^\sigma{}_{[\mu}(\Phi_{\nu]}D_\sigma\lambda - \Phi_\sigma D_{\nu]}\lambda)^-\right)^+.
\end{aligned} \tag{1.217}$$

1.5.5 Unification with $H_{\mathbb{G}}(\mathbb{X})$

All together, the twist Weyl multiplet coupled to the twisted vector multiplet on the symmetric gravitino background gives us the transformation laws of

$$d_{\mathrm{t}} g_{\mu\nu} = \Psi_{\mu\nu}, \tag{1.218}$$

$$d_{\mathrm{t}} \Psi_{\mu\nu} = \nabla_\mu \Phi_\nu + \nabla_\nu \Phi_\mu, \tag{1.219}$$

$$d_{\mathrm{t}} \Phi^\mu = 0, \tag{1.220}$$

$$d_{\mathrm{t}} A_\mu = \psi_\mu + \Phi^\sigma(\chi_{\sigma\mu} - \Psi_{\sigma\mu}\lambda - g_{\sigma\mu}\eta), \tag{1.221}$$

$$d_{\mathrm{t}} \psi_\mu = -D_\mu \phi + \Phi^\sigma \left(F^-_{\sigma\mu} + D_{\sigma\mu} - g_{\sigma\mu}[\lambda, \phi] + 2\lambda(\nabla_{[\sigma}\Phi_{\nu]})^- + ({\Psi^\rho}_{[\sigma}\chi_{\mu]\rho})^-\right), \tag{1.222}$$

$$d_{\mathrm{t}} \phi = -\Phi^\sigma \psi_\sigma, \tag{1.223}$$

$$d_{\mathrm{t}} \lambda = \eta, \tag{1.224}$$

$$d_{\mathrm{t}} \eta = [\phi, \lambda] + \Phi^\sigma D_\sigma \lambda, \tag{1.225}$$

$$d_{\mathrm{t}} \chi_{\mu\nu} = F^+_{\mu\nu} - D_{\mu\nu} + 2\lambda(\nabla_{[\mu}\Phi_{\nu]})^+ - 4(\Phi_{[\mu}D_{\nu]}\lambda)^+ - ({\Psi^\sigma}_{[\mu}\chi_{\nu]\sigma})^-, \tag{1.226}$$

$$\begin{aligned} d_{\mathrm{t}} D_{\mu\nu} = {} & 2(D_{[\mu}\psi_{\nu]})^+ - [\phi, \chi_{\mu\nu}] - ({\Psi^\sigma}_{[\mu}D_{\nu]\sigma})^- + 2(\Phi_{[\mu}D^\sigma\chi_{\nu]\sigma})^+ - 2(\Phi_{[\mu}D_{\nu]}\eta)^+ \\ & + \left({\Psi^\sigma}_{[\mu}\left(F^-_{\nu]\sigma} + \lambda(\nabla_{\nu]}\Phi_\sigma - \nabla_\sigma\Phi_{\nu]})^- - ({\Psi^\rho}_{[\sigma}\chi_{\nu]]\rho})^-\right)\right)^+ - 4(\Phi_{[\mu}[\psi_{\nu]}, \lambda])^+ \\ & - 2\Phi^\sigma(\Psi_{\sigma[\mu}D_{\nu]}\lambda)^+ - 2\left({\Psi^\sigma}_{[\mu}(\Phi_{\nu]}D_\sigma\lambda - \Phi_\sigma D_{\nu]}\lambda)^-\right)^+ . \end{aligned} \tag{1.227}$$

Careful computation reveals that this differential squares to

$$d_{\mathrm{t}}^2 = \mathcal{L}^{(A)}_\Phi + \delta_{\phi + \Phi^\sigma\Phi_\sigma\lambda}. \tag{1.228}$$

Therefore, we have a differential that appears to be equivariant with respect gauge transformations and diffeomorphisms in nearly the same fashion as $\mathbb{Q}$, just with a different gauge transformation parameter. The similarities between (1.218)-(1.227) and (1.114)-(1.120) are certainly wanting for an explanation. The answer? Namely,

$$\mathbb{Q} = d_{\mathrm{t}}. \tag{1.229}$$

Accepting this, we do away with d_{t} and always write $\mathbb{Q}$. In our view, the differences are simple matter of field redefintions of (A, ψ, D), and thus we will hence forth write these fields as they appear in twisted supergravity and in (1.218)-(1.227) with a superscript "t." Note that this also requires us to write a similar superscript on the field strength F_A and all gauge covariant derivatives D_A.

We will break our redefintions into two parts, first redefining the gauge field A along with the auxiliary field D, and then redefining ψ. We take this perspective in order to stress the point that each of these maneuvers is independent and might individually provide insight into the presentation preferred by supergravity. An alternatively and perhaps more revealing approach that the difference between our two presentations of the Cartan model of $H_{\mathbb{G}}(\mathbb{X})$ is the result of a different splitting of the action of $\mathbb{G}$ on the vector multiplet fields. Thereby considering each redefinition independently, we draw focus on each particular changes in the group action.

Our first field redefintion is given by

$$A^{\mathrm{t}}_{\mu} \longrightarrow A_{\mu} = A^{t}_{\mu} + \Phi_{\mu}\lambda, \tag{1.230}$$

$$D^{\mathrm{t}}_{\mu\nu} \longrightarrow D_{\mu\nu} = D^{\mathrm{t}}_{\mu\nu} + 2(\Phi_{[\mu}D_{\nu]}\lambda)^{+}. \tag{1.231}$$

Crucially, the first shift has metric dependence in the lowered index of Φ_{μ}. As previously mentioned, this incurs a change in both the field strength and the gauge covariant derivative given by

$$F^{\mathrm{t}}_{\mu\nu} = F_{\mu\nu} - 2\lambda(\nabla_{[\mu}\Phi_{\nu]}) + 2\Phi_{[\mu}D_{\nu]}\lambda, \tag{1.232}$$

$$D^{\mathrm{t}}_{\mu}\mathcal{O} = D_{\mu}\mathcal{O} - \Phi_{\mu}[\lambda, \mathcal{O}], \tag{1.233}$$

where $\mathcal{O}$ is any adjoint valued field. In particular, we have $D^{\mathrm{t}}_{A}\lambda = D_{A}\lambda$. The resulting transformations are then given by

$$\mathbb{Q}g_{\mu\nu} = \Psi_{\mu\nu}, \tag{1.234}$$

$$\mathbb{Q}\Psi_{\mu\nu} = \nabla_{\mu}\Phi_{\nu} + \nabla_{\nu}\Phi_{\mu}, \tag{1.235}$$

$$\mathbb{Q}\Phi^{\mu} = 0, \tag{1.236}$$

$$\mathbb{Q}A_{\mu} = \psi^{\mathrm{t}}_{\mu} + \Phi^{\rho}\chi_{\rho\mu}, \tag{1.237}$$

$$\mathbb{Q}\psi_{\mu} = -D_{\mu}\phi + \Phi^{\rho}\left(F^{-}_{\rho\mu} + D_{\rho\mu} + (\Psi^{\sigma}{}_{[\rho}\chi_{\mu]\sigma})^{-}\right), \tag{1.238}$$

$$\mathbb{Q}\phi = -\Phi^{\rho}\psi^{\mathrm{t}}_{\rho}, \tag{1.239}$$

$$\mathbb{Q}\lambda = \eta, \tag{1.240}$$

$$\mathbb{Q}\eta = [\phi, \lambda] + \Phi^{\rho}D_{\rho}\lambda, \tag{1.241}$$

$$\mathbb{Q}\chi_{\mu\nu} = F^{+}_{\mu\nu} - D_{\mu\nu} - (\Psi^{\rho}{}_{[\mu}\chi_{\nu]\sigma})^{-}, \tag{1.242}$$

$$\begin{aligned}\mathbb{Q}D_{\mu\nu} = {} & (D_{\mu}\psi^{\mathrm{t}}_{\nu} - D_{\nu}\psi^{\mathrm{t}}_{\mu})^{+} - [\phi, \chi_{\mu\nu}] + 2(\Phi_{[\mu}D_{\rho}\chi^{\rho}_{\nu]})^{+} \\ & - (\Psi^{\rho}{}_{[\mu}D_{\nu]\sigma})^{-} + \left(\Psi^{\rho}{}_{[\mu}(F^{-}_{\nu]\rho} - (\Psi^{\sigma}{}_{[\rho}\chi_{\nu]]\sigma})^{-}\right)^{+}.\end{aligned} \tag{1.243}$$

Note that the shift (1.230) has already solved the mystery of the extra gauge transformation in d^2_{t} of (1.228) above. We now have

$$\mathbb{Q}^2 = \mathcal{L}^{(A)}_{\Phi} + \delta_{\phi} = \mathcal{L}^{(A^{\mathrm{t}})}_{\Phi} + \delta_{\phi + \Phi^{\sigma}\Phi_{\sigma}\lambda}. \tag{1.244}$$

Another feature of this shift is that it eliminates all of the derivatives on Φ from our transformation laws. When we turn to the construction of our action, this means that we can avoid introducing any potential kinetic-like terms for the background Φ field, which would be of assistance in the unexplored case of a dynamical Φ.

Next, we encounter the vector gaugino redefinition

$$\psi^{\mathrm{t}}_\mu \longrightarrow \psi_\mu = \psi^{\mathrm{t}}_\mu + \Phi^\rho \chi_{\rho\mu}, \tag{1.245}$$

This brings us exactly back to the transformations (1.114)-(1.118) and (1.122)-(1.123), which we repeat here for the benefit of the reader.

$$\mathbb{Q} g_{\mu\nu} = \Psi_{\mu\nu}, \qquad \mathbb{Q} A_\mu = \psi_\mu, \tag{1.246}$$

$$\mathbb{Q}\Psi_{\mu\nu} = \nabla_\mu \Phi_\nu + \nabla_\nu \Phi_\mu, \qquad \mathbb{Q}\psi_\mu = -D_\mu \phi + \Phi^\sigma F_{\sigma\mu} \tag{1.247}$$

$$\mathbb{Q}\Phi^\sigma = 0, \qquad \mathbb{Q}\phi = -\Phi^\sigma \psi_\sigma, \tag{1.248}$$

$$\mathbb{Q}\lambda = \eta, \tag{1.249}$$

$$\mathbb{Q}\eta = [\phi, \lambda] + \Phi^\sigma D_\sigma \lambda, \tag{1.250}$$

$$\mathbb{Q}\chi_{\mu\nu} = F^+_{\mu\nu} - D_{\mu\nu} - (\Psi^\sigma{}_{[\mu}\chi_{\nu]\sigma})^-, \tag{1.251}$$

$$\begin{aligned}\mathbb{Q} D_{\mu\nu} = {} & 2(D_{[\mu}\psi_{\nu]})^+ - [\phi, \chi_{\mu\nu}] - \Phi^\sigma D_\sigma \chi_{\mu\nu} - ((\nabla_\mu \Phi^\sigma)\chi_{\sigma\nu} - (\nabla_\nu \Phi^\sigma)\chi_{\sigma\mu})^+ \\ & + (\Psi^\sigma{}_{[\mu} F^-_{\nu]\sigma})^+ - (\Psi^\sigma{}_{[\mu} D_{\nu]\sigma})^- + \frac{1}{2}\Psi^{\rho\sigma}\Psi_{\rho[\mu}\chi_{\nu]\sigma}\end{aligned}$$

Hence, we come to the grand conclusion that twisted and truncated $\mathcal{N} = 2$ Eulcidean supergravity on a symmetric gravitino background is a presentation of the base of Cartan model of equivariant cohomology on $\mathcal{A} \times \mathsf{Met}(\mathbb{X})$ with respect to $\mathcal{G} \times \mathsf{Diff}_+(\mathbb{X})$ and two anti-ghost modules! This is one of the major results of this work and perfectly exemplifies the spirit of physical mathematics.

Now, with that out of the way, let's get back to the action!

2 The Action

2.1 Overview

The action of Donaldson-Witten theory S_{UV} for an arbitrary Lie group G in (0.92) is a scalar functional of the twisted vector multiplet field content which is $\mathcal{Q}$-closed and a scalar under the Euclidean group. We can write it as a $\mathcal{Q}$-exact piece plus a topological term, namely

$$S_{\mathrm{UV}} = \mathcal{Q}V_{\mathrm{UV}} + \frac{i\tau_0}{8\pi}\int_{\mathbb{X}} \mathrm{Tr} F_A \wedge F_A. \tag{2.1}$$

Likewise, the IR action with an arbitrary prepotential $\mathcal{F}$ and an $G = \mathrm{U}(1)$ group can be written

$$S_{\mathrm{IR}} = \mathcal{Q}(V_{\mathrm{IR}} + \overline{V}_{\mathrm{IR}}) + \mathsf{C}_{\mathrm{IR}}. \tag{2.2}$$

Our goal in this section will be to construct a new action $\mathbb{S}$ which is not $\mathcal{Q}$-closed, but $\mathbb{Q}$-closed. Further, when we turn off the background supergravity fields Ψ and Φ, we want this action to reduce to either S_{UV} in the case of a quadratic potential or S_{IR} in the case of $G = \mathrm{U}(1)$. This can be summarized by the three conditions[40]

$$\mathbb{Q}\mathbb{S} = 0, \tag{2.3}$$

$$\mathbb{S}\Big|_{\mathcal{F}=\frac{1}{2}\tau_0\mathrm{Tr}[\phi^2],\overline{\mathcal{F}}=\frac{1}{2}\overline{\tau}_0\mathrm{Tr}[\lambda^2],\Psi,\Phi=0} = S_{\mathrm{UV}}, \tag{2.4}$$

$$\mathbb{S}\Big|_{G=\mathrm{U}(1),\Psi,\Phi=0} = S_{\mathrm{IR}}. \tag{2.5}$$

[40] We note that (2.4) and (2.5) are technically mutually incompatible due to an overall scale factor in the normalization between S_{UV} and S_{IR}. Nevertheless, the structure of the terms do allow our conditions to be meaningful. For posterity, we choose to align with S_{IR}.

These requirements are tantamount to identifying a coupling of the super Yang Mills theory to our twisted supergravity background Weyl multiplet. We will first present an action $\mathbb{S}$ which minimally satisfies these conditions by generalizing the primitives $V_{\mathrm{UV}}, V_{\mathrm{IR}}$, and $\overline{V}_{\mathrm{IR}}$. Then, in our final excursus, we will do a lightening fast and astoundingly cursory review of superconformal tensor calculus and present a second action $\mathbb{S}^{\mathrm{t}}$ which satisfies (2.3)-(2.5). We will conclude this section by showing that $\mathbb{S} = \mathbb{S}^{\mathrm{t}} + \mathbb{Q}(\overline{\mathbb{A}} + \mathbb{A})$, that is, the two actions only differ by cohomologically uninteresting $\mathbb{Q}$-exact terms.

2.2 Minimal Action

We define

$$\mathbb{S} = \mathbb{Q}(\mathbb{V} + \overline{\mathbb{V}}) + \mathsf{C}, \tag{2.6}$$

where

$$\begin{aligned}\mathbb{V} = \frac{i}{2^4\pi}\int_{\mathbb{X}} d^4x\sqrt{g}\,\Big[&-\frac{1}{2}\mathcal{F}_{IJ}(F^{+,I}_{\mu\nu} + D^I_{\mu\nu})\chi^{\mu\nu,J} - 2\mathcal{F}_{IJ}\psi^I_\sigma D^\sigma\lambda^J \\ &+ \mathcal{F}_{IJK}\psi^I_\mu\psi^J_\nu\chi^{\mu\nu,K} - 2\mathcal{F}_I[\lambda,\eta]^I\Big],\end{aligned} \tag{2.7}$$

$$\begin{aligned}\overline{\mathbb{V}} = \frac{i}{2^4\pi}\int_{\mathbb{X}} d^4x\sqrt{g}\,\Big[&\frac{1}{2}\overline{\mathcal{F}}_{IJ}(F^{+,I}_{\mu\nu} + D^I_{\mu\nu})\chi^{\mu\nu,J} - 2\overline{\mathcal{F}}_I D_\sigma\psi^{\sigma,I} \\ &+ \frac{i}{12}\overline{\mathcal{F}}_{IJK}\chi_\mu{}^{\rho,I}\chi^{\mu\sigma,J}\chi^K_{\rho\sigma} - 2\overline{\mathcal{F}}_I[\phi,\eta]^I\Big],\end{aligned} \tag{2.8}$$

and

$$\mathsf{C} = \frac{i}{2^4\pi}\int_{\mathbb{X}} d^4x \left[\frac{1}{4}\mathcal{F}_{IJ}\epsilon^{\mu\nu\rho\sigma}F^I_{\mu\nu}F^J_{\rho\sigma} - \frac{1}{2}\mathcal{F}_{IJK}\epsilon^{\mu\nu\rho\sigma}\psi^I_\mu\psi^J_\nu F^K_{\rho\sigma} + \frac{1}{12}\mathcal{F}_{IJKL}\epsilon^{\mu\nu\rho\sigma}\psi^I_\mu\psi^J_\nu\psi^K_\rho\psi^L_\sigma\right]. \tag{2.9}$$

Here, we use the indices I, J, K, L to denote the gauge indices. Multiple indices on a prepotential indicate derivatives by the respective scalar. In the case of non-abelian gauge groups they serve as indices for the generators of the adjoint representation and, in the IR they label different abelian vector multiplets. Thus, for example, for a single abelian vector multiplet, where the indices can be dropped, we would write

$$\mathcal{F}_{IJ} = \mathcal{F}_{11} = \frac{\partial^2\mathcal{F}}{\partial\phi^2} = \tau \qquad\qquad \overline{\mathcal{F}}_{IJ} = \overline{\mathcal{F}}_{11} = \frac{\partial^2\overline{\mathcal{F}}}{\partial\lambda^2} = \overline{\tau} \tag{2.10}$$

Let us check that $\mathbb{S}$ satisfies the first of our conditions of $\mathbb{Q}\mathbb{S} = 0$. Since $\mathbb{Q}^2 = \mathcal{L}^{(A)}_\Phi + \delta_\phi$ and both $\overline{\mathbb{V}}$ and $\mathbb{V}$ are scalars in the trivial representation of G, it is clear that $\mathbb{Q}(\overline{\mathbb{V}} + \mathbb{V}) = 0$. For the non-exact piece, we have

$$\begin{aligned}
\mathbb{Q}\mathsf{C} &= \mathcal{Q}\mathsf{C} + \mathsf{K}\mathsf{C} \\
&= \frac{i}{2^4\pi}\int_{\mathbb{X}} d^4x \left[\nabla_\mu\left(\mathcal{F}_{IJ}\epsilon^{\mu\nu\alpha\beta}\psi^I_\nu F^J_{\alpha\beta} - \frac{1}{3}\mathcal{F}_{IJK}\epsilon^{\mu\nu\alpha\beta}\psi^I_\nu\psi^J_\alpha\psi^K_\beta\right)\right] \\
&\quad + \frac{i}{2^4\pi}\int_{\mathbb{X}} d^4x\Phi^\sigma\epsilon^{\mu\nu\alpha\beta}\Big[-\frac{1}{4}\mathcal{F}_{IJK}(\psi^I_\sigma F^J_{\mu\nu}F^K_{\alpha\beta} - 4\psi^I_\mu F^J_{\sigma\nu}F^K_{\alpha\beta}) \\
&\qquad\qquad + \frac{1}{6}\mathcal{F}_{IJKL}(3\psi^I_\sigma\psi^J_\mu\psi^K_\nu F^L_{\alpha\beta} - 2\psi^I_\mu\psi^J_\nu\psi^K_\alpha F^L_{\sigma\beta}) \\
&\qquad\qquad - \frac{1}{12}\mathcal{F}_{IJKLM}\psi^I_\sigma\psi^J_\mu\psi^K_\nu\psi^L_\alpha\psi^M_\beta\Big] \\
&= \frac{i}{2^4\pi}\int \mathrm{Tr}\Big[\iota_\Phi\Big(-\mathcal{F}_{IJK}\psi^I\wedge F^J\wedge F^K + \frac{1}{3}\mathcal{F}_{IJKL}\psi^I\wedge\psi^J\wedge\psi^K\wedge F^L \\
&\qquad\qquad - \frac{1}{60}\mathcal{F}_{IJKLM}\psi^I\wedge\psi^J\wedge\psi^K\wedge\psi^L\wedge\psi^M\Big)\Big].
\end{aligned}$$

Since we are working on a four manifold, there is no support for the five forms in the final line above, and we conclude that $\mathbb{Q}\mathsf{C} = 0$. Hence, we see that $\mathbb{S}$ is indeed $\mathbb{Q}$-closed as desired.

Turning to our two other conditions, we need to look at the explicit form of $\mathbb{S}$. Computing the action of $\mathbb{Q}$ in (2.6), we have

$$
\begin{aligned}
\mathbb{S} = \mathbb{S}_0 - \frac{1}{2}\int_{\mathbb{X}} d^4x\sqrt{g}\Psi^{\mu\nu}(\Lambda_{\mu\nu} + \overline{\Lambda}_{\mu\nu}) \\
+ \int_{\mathbb{X}} d^4x\sqrt{g}\Phi^{\sigma}(Z_\sigma + \overline{Z}_\sigma) + \int_{\mathbb{X}} d^4x\sqrt{g}\Psi^{\mu\sigma}\Psi^{\nu}{}_{\sigma}(\overline{\Upsilon}_{\mu\nu} + \Upsilon_{\mu\nu}).
\end{aligned}
\tag{2.11}
$$

Here,

$$
\begin{aligned}
\mathbb{S}_0 &= \mathcal{Q}(\overline{\mathbb{V}} + \mathbb{V}) + \mathsf{C} \\
&= \frac{i}{2^4\pi}\int_{\mathbb{X}} d^4x\sqrt{g}\Bigg[\frac{1}{2}\overline{\mathcal{F}}_{IJ}F^{+\,I}_{\mu\nu}F^{\mu\nu,J}_{+} - \frac{1}{2}\mathcal{F}_{IJ}F^{-\,I}_{\mu\nu}F^{\mu\nu,J}_{-} + 4i\mathrm{Im}\mathcal{F}_{IJ}D_\sigma\phi^I D^\sigma\lambda^J \\
&\qquad +i\mathrm{Im}\mathcal{F}_{IJ}D_{\mu\nu}{}^{I}D^{\mu\nu,J} + 2\mathcal{F}_{IJ}\psi_\sigma{}^{I}D^\sigma\eta^J - 2\overline{\mathcal{F}}_{IJ}\eta^I D_\sigma\psi^{\sigma,J} \\
&\qquad -2\mathcal{F}_{IJ}\psi_\mu{}^{I}(D_\nu\chi^{\mu\nu})^J + 2\overline{\mathcal{F}}_{IJ}(D_{[\mu}\psi_{\nu]}{}^{I})^+\chi^{\mu\nu,J} \\
&\qquad +\frac{1}{2}\overline{\mathcal{F}}_{IJK}\eta^I(F^{+,J}_{\mu\nu} + D_{\mu\nu}{}^{J})\chi^{\mu\nu,K} + \\
&\qquad +\mathcal{F}_{IJK}\psi_\mu{}^{I}\psi_\nu{}^{J}(F^{\mu\nu,K}_{-} - D^{\mu\nu,K}) \\
&\qquad +\frac{1}{12}\sqrt{g}^{-1}\mathcal{F}_{IJKL}\epsilon^{\mu\nu\rho\sigma}\psi_\mu{}^{I}\psi_\nu{}^{J}\psi_\rho{}^{K}\psi_\sigma{}^{L} \\
&\qquad +\frac{i}{12}\overline{\mathcal{F}}_{IJKL}\eta^I\chi_\mu{}^{\rho,J}\chi^{\mu\sigma,K}\chi_{\rho\sigma}{}^{L} \\
&\qquad -\frac{i}{2}\overline{\mathcal{F}}_{IJK}(F^{+\,I}_{\mu\rho} - D_{\mu\rho}{}^{I})\chi^{\mu\sigma,J}\chi_\sigma{}^{\rho,K} \\
&\qquad + i\mathrm{Im}\mathcal{F}_{IJ}[\phi,\chi_{\mu\nu}]^I\chi^{\mu\nu,J} + 2\mathcal{F}_{IJ}\psi_\sigma{}^{I}[\psi^\sigma,\lambda]^J - 2\overline{\mathcal{F}}_I[\psi_\sigma,\psi^\sigma]^I \\
&\qquad - 2\mathcal{F}_I[\eta,\eta]^I - 2\overline{\mathcal{F}}_{IJ}\eta^I[\phi,\eta]^J - 2\mathcal{F}_I[\lambda,[\phi,\lambda]]^I - 2\overline{\mathcal{F}}_I[\phi,[\phi,\lambda]^I\Bigg],
\end{aligned}
\tag{2.12}
$$

$$\Lambda_{\mu\nu} = \frac{i}{2^4\pi}\left[-\mathcal{F}_{IJ}F^{-,I}_{\mu\sigma}\chi_\nu{}^{\sigma,J} - \mathcal{F}_{IJ}F^{-,I}_{\nu\sigma}\chi_\mu{}^{\sigma,J} + \mathcal{F}_{IJK}(\psi^I_\mu\psi^J_\sigma)^-\chi_\nu{}^{\sigma,K}\right.$$
$$+\mathcal{F}_{IJK}(\psi^I_\nu\psi^J_\sigma)^-\chi_\mu{}^{\sigma,K} + 2g_{\mu\nu}\mathcal{F}_{IJ}\psi_\sigma{}^I D^\sigma\lambda^J$$
$$\left.-2\mathcal{F}_{IJ}\psi_\mu{}^I D_\nu\lambda^J - 2\mathcal{F}_{IJ}\psi_\nu{}^I D_\mu\lambda^J + 2g_{\mu\nu}\mathcal{F}_I[\lambda,\eta]^I\right], \tag{2.13}$$

$$\overline{\Lambda}_{\mu\nu} = \frac{i}{2^4\pi}\left[\overline{\mathcal{F}}_{IJ}F^{-I}_{\mu\sigma}\chi_\nu{}^{\sigma,J} + \overline{\mathcal{F}}_{IJ}F^{-,I}_{\nu\sigma}\chi_\mu{}^{\sigma,J} - 2\overline{\mathcal{F}}_I D_\mu\psi_\nu{}^I - 2\overline{\mathcal{F}}_I D_\nu\psi_\mu{}^I\right.$$
$$\left.+2g_{\mu\nu}\overline{\mathcal{F}}_I D_\sigma\psi^{\sigma,I} + g_{\mu\nu}\overline{\mathcal{F}}_I[\phi,\eta]^I + \frac{i}{12}\overline{\mathcal{F}}_{IJK}\widetilde{\mathsf{d}}_{\not\Psi}(\chi_\mu{}^{\rho,I}\chi^{\mu\sigma,J}\chi_{\rho\sigma}{}^K)\right], \tag{2.14}$$

$$Z_\sigma = \frac{i}{2^4\pi}\left[\frac{1}{2}\mathcal{F}_{IJK}\psi_\sigma{}^I(F^{+\,J}_{\mu\nu} + D_{\mu\nu}{}^J)\chi^{\mu\nu,K} + 2\mathcal{F}_{IJK}F_{\sigma\mu}{}^I\psi_\nu{}^J\chi^{\mu\nu,K}\right.$$
$$+ 2\mathcal{F}_{IJK}\psi_\sigma{}^I\psi_\rho{}^J D^\rho\lambda^K + \frac{1}{2}\mathcal{F}_{IJ}D_\sigma\chi_{\mu\nu}{}^I\chi^{\mu\nu,J} - \mathcal{F}_{IJ}D_\mu(\chi_{\sigma\nu}{}^I\chi^{\mu\sigma,J})$$
$$- \mathcal{F}_{IJK}D_\mu\phi^I\chi_{\sigma\nu}{}^J\chi^{\mu\nu,K} - 2\mathcal{F}_{IJ}F_{\sigma\rho}{}^I D^\rho\lambda^J + 2\mathcal{F}_{IJ}\psi_\sigma{}^I[\lambda,\eta]^J$$
$$\left.- \mathcal{F}_{IJKL}\psi_\sigma{}^I\psi_\mu{}^J\psi_\nu{}^K\chi^{\mu\nu,L} - 2\mathcal{F}_I[\lambda, D_\sigma\lambda]^I\right], \tag{2.15}$$

$$\overline{Z}_\sigma = \frac{i}{2^4\pi}\left[-\frac{1}{2}\overline{\mathcal{F}}_{IJ}D_\sigma\chi_{\mu\nu}{}^I\chi^{\mu\nu,J} + \overline{\mathcal{F}}_{IJ}D_\mu(\chi^I_{\sigma\nu}\chi^{\mu\nu,J}) + \frac{i}{12}\overline{\mathcal{F}}_{IJK}D_\mu\lambda^I\chi_{\sigma\nu}{}^J\chi^{\mu\nu,K}\right.$$
$$\left.+ 2\overline{\mathcal{F}}_{IJ}F_{\sigma\rho}{}^I D^\rho\lambda^J + 2\overline{\mathcal{F}}_I[\psi_\sigma,\eta]^I - 2\overline{\mathcal{F}}_I[\phi, D_\sigma\lambda]^I\right], \tag{2.16}$$

$$\overline{\Upsilon}_{\mu\nu} = -\frac{i}{2^6\pi}\overline{\mathcal{F}}_{IJ}\chi_{\mu\rho}{}^I\chi_\nu{}^{\rho,J}, \tag{2.17}$$

$$\Upsilon_{\mu\nu} = \frac{i}{2^6\pi}\mathcal{F}_{IJ}\chi_{\mu\rho}{}^I\chi_\nu{}^{\rho,J}. \tag{2.18}$$

Here, to avoid needless variations of self-dual fields, we have written the last term of (2.14) as $\widetilde{\mathsf{d}}_{\not\Psi}(\cdots)$, which denotes the variation, with the gravitino extracted to the left. Now, comparing $\mathbb{S}_0$ to S_{UV} and S_{IR}, it is clear that we have the desired alignment of (2.4) and (2.5). This can even be seen at the level of the primitives, as

$$(\mathbb{V}+\overline{\mathbb{V}})\big|_{\mathcal{F}=\frac{1}{2}\tau_0\mathrm{Tr}[\phi^2],\overline{\mathcal{F}}=\frac{1}{2}\overline{\tau}_0\mathrm{Tr}[\lambda^2],\Psi,\Phi=0}=V_{\mathrm{UV}}, \tag{2.19}$$

$$(\mathbb{V}+\overline{\mathbb{V}})\big|_{G=\mathrm{U}(1),\Psi,\Phi=0}=\overline{V}_{\mathrm{IR}}+V_{\mathrm{IR}}. \tag{2.20}$$

Therefore we declare success! We also mention that $\mathbb{S}$ is as minimal a coupling to supergravity as possible given our desiderata, as it has the exact same non-exact term and we are essentially conducting a "twisted supergravity completion" of $\mathcal{Q}(\mathbb{V}+\overline{\mathbb{V}})$ with $\mathbb{Q}(\mathbb{V}+\overline{\mathbb{V}})$.

Finally, before we turn to the second generalized action, we define the UV and IR limits of $\mathbb{S}$ with

$$\mathbb{S}_{\mathrm{UV}}=\mathbb{S}\big|_{\mathcal{F}=\frac{1}{2}\tau_0\mathrm{Tr}[\phi^2],\overline{\mathcal{F}}=\frac{1}{2}\overline{\tau}_0\mathrm{Tr}[\lambda^2],\Psi,\Phi=0}, \tag{2.21}$$

$$\mathbb{S}_{\mathrm{IR}}=\mathbb{S}\big|_{G=\mathrm{U}(1),\Psi,\Phi=0}. \tag{2.22}$$

2.3 *Excursus:* Superconformal Tensor Calculus Action

In Section 0.2.3 we saw that the constraint of $\mathcal{N}=2$ supersymmetry led to a single general formula for the action of a single vector multiplet. With even more symmetry on the table, there is a similar formula for the action of $\mathcal{N}=2$ supergravity known as the *chiral density formula.* Unfortunately, it is not generally written in terms of the vector multiplet representation, but rather the superconformal chiral and anti-chiral multiplet. Thankfully, through the method of *superconformal tensor calculus,* one can constrain these multiplets to arrive directly at the fields of the vector multiplet, giving one a general action principle for a $\mathcal{N}=2$ supergravity vector multiplet. After a twist and a push onto our symmetric gravitino background, we then arrive at a second generalized action which satisfies (2.3)-(2.5). In this final excursus, we will take a brief foray into the chiral and antichiral multiplets and then present the resulting action $\mathbb{S}^{\mathrm{t}}$. We will then, after conducting the previously mentioned field redefinition of Section 1.5.5, show that this action differs from $\mathbb{S}$ by a $\mathbb{Q}$-exact term.

2.3.1 The Construction

The superconformal (anti) chiral multiplet of conformal supergravity is defined as the representation of the conformal superalgebra whose bottom component is a scalar field that transforms into a (anti) chiral spinor under a supersymmetric variation. The remaining fields of the multiplet are simply a result of the supersymmetric completion of the multiplet. Prior to the twist, the $\mathcal{N} = 2$ chiral multiplet is built from a real scalar A_+, a left-handed spinor G^i_+, a symmetric $\mathfrak{su}(2)_+$ field $\mathsf{B}^{(ij)}$, a self-dual two form field F^+_{ab}, a left-handed spinor Λ^i_+, and a real scalar C_+. The $\mathcal{N} = 2$ anti-chiral multiplet mirrors these fields, with each $+$ replaced by a $-$ and the word "left" reflected over to "right." Both multiplets have eight bosonic and fermionic degrees of freedom.

After the twist, we can identify the $\mathfrak{su}(2)_{\mathrm{R}}$ indices with the $\mathfrak{su}(2)_+$ and we find that, as in the case of the vector multiplet fields, all spinors either split into a fermionic zero for and a fermionic self-dual two form or become a fermionic 1-form. We can collect the two multiplets as

$$(\mathsf{A}_+, \mathsf{G}, \mathsf{G}_{\mu\nu}, \mathsf{B}_{+\mu\nu}, \mathsf{F}^+_{\mu\nu}, \Lambda, \Lambda_{\mu\nu}, \mathsf{C}_+), \tag{2.23}$$

and

$$(\mathsf{A}_-, \mathsf{G}_\mu, \mathsf{B}_{-\mu\nu}, \mathsf{F}^-_{\mu\nu}, \Lambda_\mu, \mathsf{C}_-). \tag{2.24}$$

Here, all fields labeled by $\mathsf{A}, \mathsf{B}, \mathsf{C}$, or F are bosonic, and the rest fermionic. In addition, $\mathsf{F}^-_{\mu\nu}$ is anti-self-dual, while $\mathsf{G}_{\mu\nu}, \mathsf{F}^+_{\mu\nu}, \mathsf{B}_{+\mu\nu}$, and, unfortunately for notation, $\mathsf{B}_{-\mu\nu}$ are all self-dual.

These fields each have explicit and rather complicated transformation laws which do not mix amounst each other, but do involve the fields of the twisted Weyl multiplet. We will, as now should be expected, refer the reader to [12], for the full transformations, but do crucially note, for reasons that soon become clear, that C_- transformations into a total derivative plus a term of the form $-\frac{1}{2}\Psi^\sigma{}_\sigma\mathsf{C}_-$ and that the transformation of Λ contains $\frac{i}{\sqrt{2}}\mathsf{C}_+$.

The action for the chiral or anti-chiral multiplet has a fixed form, dictated by the chiral density formula. After the twist and truncation, but before moving to a symmetric gravitino background, it leds us to the action

$$\mathbb{S}_{\mathrm{CDF}} = \int_{\mathbb{X}} d^4x\sqrt{g}\left[\mathrm{L}_+ + \mathrm{L}_-\right], \tag{2.25}$$

with

$$\begin{aligned}
\mathrm{L}_+ = {} & \mathsf{C}_+ - \frac{i}{\sqrt{2}}\Psi_\rho{}^\rho\Lambda + 4\sqrt{2}T^-_{\mu\nu}\Psi^\mu{}_\rho\mathsf{G}^{\rho\nu} - 4\mathsf{A}_+T^-_{\mu\nu}T^{\mu\nu-} \\
& - i\Psi_\mu{}^\rho\Psi_{\nu\rho}(\mathsf{F}^{\mu\nu+} + B^{\mu\nu}_+) - 4\mathsf{A}_+T^{\mu\nu-}\Psi_\mu{}^\rho\Psi_{\nu\rho} \\
& - \sqrt{2}\sqrt{g}^{-1}\epsilon^{\mu\nu\rho\sigma}\Psi_\mu{}^\lambda\Psi_{\nu\lambda}\Psi_\rho{}^\delta\mathsf{G}_{\delta\sigma} + \frac{1}{2}\sqrt{g}^{-1}\epsilon^{\mu\nu\rho\sigma}\Psi_\mu{}^\lambda\Psi_{\nu\lambda}\Psi_\rho{}^\delta\Psi_{\sigma\delta}\mathsf{A}_+,
\end{aligned} \tag{2.26}$$

$$\mathrm{L}_- = \mathsf{C}_-. \tag{2.27}$$

We note that the two terms in the chiral density formula (2.25) are actually independent and we have chosen their relative coefficient by hand. Importantly, the anti chiral multiplet density is simply C_- due to our truncation, and given our earlier comment about its transformation, it is clear that it vanishes under integration, as

$\mathbb{Q}(\sqrt{g}\mathsf{C}_-)$ is a total derivative. In addition, we see that $\mathbb{Q}(-i\sqrt{2}\sqrt{g}\Lambda)$ contains the first two terms of $\sqrt{g}\mathsf{L}_+$. Together, these two observations suggest that a minimal $\mathbb{Q}$-closed action that necessarily contains all gravity degree zero terms would take the form

$$\mathbb{S}^{\rm t} = \int_{\mathbb{X}} d^4x \, [\sqrt{g}\mathsf{C}_- + \mathbb{Q}(\sqrt{g}\Lambda)] \,, \tag{2.28}$$

where we have put our hand back in to change the coefficient of the second term. Lamentably, this action is still written in terms of the twisted chiral and twisted anti chiral multiplet fields. In order to convert this to an action of the twisted vector multiplet and twisted Weyl multiplet fields, we must partake in a hearty helping of superconformal tensor calculus. Weighty jargon aside, for the case in front of us, this means identifying A_+ of the twisted chiral multiplet with our prepotential $\overline{\mathcal{F}}$ and likewise A_- of the twisted anti chiral multiplet with our prepotential $\mathcal{F}$. Then, maintaining $\mathbb{Q}\mathsf{A}_+ = \mathbb{Q}\overline{\mathcal{F}}$ and $\mathbb{Q}\mathsf{A}_- = \mathbb{Q}\mathcal{F}$mandates a further identification of the G fields with vector multiplet fields, and so forth and so on into the cascade of consistency conditions, until we end up with an expression for each field of (2.23) and (2.24) in terms of exclusively twisted vector multiplet and twist Weyl multiplet fields. This process is done with the "t" fields of the original transformation laws of twisted supergravity, related the those of the Cartan model through (1.230)-(1.231)+(1.245). We will therefore make use of the "t" superscript on our fields.

All told, after a light rescaling, we arrive at the following expressions

$$\begin{aligned}\Lambda^{\rm t} = \frac{i}{2^4\pi}\Bigg[&\frac{1}{2}\overline{\mathcal{F}}_{IJ}(\widehat{\mathbf{F}}^{{\rm t}+,I}_{\mu\nu} + D^{{\rm t}I}_{\mu\nu})\chi^{\mu\nu,J} - 2\overline{\mathcal{F}}_I\mathbf{D}^{\rm t}_\sigma\psi^{{\rm t}\sigma,I} \\ &+ \frac{i}{12}\overline{\mathcal{F}}_{IJK}\chi_\mu{}^{\rho,I}\chi^{\mu\sigma,J}\chi^K_{\rho\sigma} - 2\overline{\mathcal{F}}_I[\phi,\eta]^I\Bigg],\end{aligned} \tag{2.29}$$

$$\mathsf{C}^{\mathrm{t}}_{-}=\frac{i}{2^4\pi}\Big[-\frac{1}{2}\mathcal{F}_{IJ}(\widehat{\mathbf{F}}^{\mathrm{t}-I}_{\mu\nu}+\lambda^I T^-_{\mu\nu})(\widehat{\mathbf{F}}^{\mathrm{t}\mu\nu-,J}+\lambda^J T^{\mu\nu-})+\frac{1}{2}\mathcal{F}_{IJ}D^I_{\mu\nu}D^{\mu\nu J}$$
$$-2\mathcal{F}_{IJ}\psi^{\mathrm{t}I}_{\mu}(\mathbf{D}^{\mathrm{t}}_{\nu}\chi^{\mu\nu})^J+2\mathcal{F}_{IJ}\psi^{\mathrm{t}I}_{\sigma}(\mathbf{D}^{\mathrm{t}\sigma}\eta)^J+2\mathcal{F}_I\mathbf{D}^{\mathrm{t}}_{\mu}\mathbf{D}^{\mathrm{t}\mu}\lambda^I$$
$$+\mathcal{F}_{IJK}\psi^{\mathrm{t}I}_{\mu}\psi^{\mathrm{t}J}_{\nu}(\widehat{\mathbf{F}}^{\mathrm{t}\mu\nu-,J}+\lambda^J T^{\mu\nu-}-D^{\mathrm{t}\mu\nu I})$$
$$+\frac{1}{12}\sqrt{g}^{-1}\mathcal{F}_{IJKL}\epsilon^{\mu\nu\rho\sigma}\psi^{\mathrm{t}I}_{\mu}\psi^{\mathrm{t}J}_{\nu}\psi^{\mathrm{t}I}_{\rho}\psi^{\mathrm{t}J}_{\sigma}+\frac{1}{2}\mathcal{F}_{IJ}[\phi,\chi_{\mu\nu}]^I\chi^{\mu\nu J}$$
$$+2\mathcal{F}_{IJ}\psi^{\mathrm{t}I}_{\mu}[\psi^{\mathrm{t}\mu},\lambda]^J-2\mathcal{F}[\eta,\eta]^I-2\mathcal{F}_I[\lambda,[\phi,\lambda]]^I\Big],\qquad(2.30)$$

where, on the symmetric gravitino background, we have

$$\mathbf{D}^{\mathrm{t}}_{\mu}\psi^{\mathrm{t}}_{\nu}=D^{\mathrm{t}}_{\mu}\psi_{\nu}-\frac{1}{2}\Psi_{\mu}{}^{\sigma}(\widehat{\mathbf{F}}^{\mathrm{t}-}_{\sigma\nu}+\lambda T^-_{\sigma\nu}+D^{\mathrm{t}}_{\sigma\nu})+\frac{1}{2}\Psi_{\mu\nu}[\lambda,\phi],\qquad(2.31)$$
$$\mathbf{D}^{\mathrm{t}}_{\mu}\phi=D^{\mathrm{t}}_{\mu}\phi+\frac{1}{2}\Psi_{\mu}{}^{\sigma}\psi^{\mathrm{t}}_{\sigma},\qquad(2.32)$$
$$\mathbf{D}^{\mathrm{t}}_{\mu}\lambda=D^{\mathrm{t}}_{\mu}\lambda,\qquad(2.33)$$
$$\mathbf{D}_{\mu}\eta=D_{\mu}\eta-\frac{1}{2}\Psi_{\mu}{}^{\sigma}D_{\sigma}\lambda-\sqrt{2}\lambda\mathrm{S}^{\sigma}{}_{[\nu\sigma]},\qquad(2.34)$$
$$\mathbf{D}^{\mathrm{t}}_{\mu}\chi_{\nu\sigma}=D^{\mathrm{t}}_{\mu}\chi_{\nu\sigma}+2(\Psi_{\mu[\nu}D^{\mathrm{t}}_{\sigma]}\lambda)^+-\sqrt{2}\lambda\mathrm{S}^+_{\mu,[\nu\sigma]},\qquad(2.35)$$
$$\widehat{\mathbf{F}}^{\mathrm{t}}_{\mu\nu}=F^{\mathrm{t}}_{\mu\nu}+\Psi_{[\mu}{}^{\sigma}\chi_{\nu]\sigma}+\frac{1}{2}\Psi_{\mu}{}^{\sigma}\Psi_{\nu\sigma}\lambda,\qquad(2.36)$$

and in particular

$$\widehat{\mathbf{F}}^{\mathrm{t}-}_{\sigma\nu}+\lambda T^-_{\sigma\nu}=F^{\mathrm{t}-}_{\mu\nu}+(\Psi_{[\mu}{}^{\sigma}\chi_{\nu]\sigma})^-+2\lambda(\nabla_{[\mu}\phi_{\nu]})^-,\qquad(2.37)$$
$$\mathbf{D}^{\mathrm{t}}_{\mu}\mathbf{D}^{\mathrm{t}\mu}\lambda=D^{\mathrm{t}}_{\mu}D^{\mathrm{t}\mu}\lambda+\frac{1}{2}\Psi^{\sigma}{}_{\sigma}[\eta,\lambda].\qquad(2.38)$$

Here, since the "t" fields are the same as our Cartan model fields at gravity degree zero, we recognize that $\mathbb{S}^{\mathrm{t}}$ as in (2.28), with (2.29) and (2.30), contains the same degree zero part as $\mathbb{S}$. Therefore, we see that $\mathbb{S}^{\mathrm{t}}$ does indeed satisfy the requirements of (2.3)-(2.5).

2.3.2 The Comparison

In this section we will show that $\mathbb{S}^{\mathrm{t}}$ is equal to $\mathbb{S}$ up to a $\mathbb{Q}$-exact term, which decouples from the theory. Therefore, these two actions formally represent the same cohomological field theory. Explicitly, we will find

$$\mathbb{S}^{\mathrm{t}} = \mathbb{S} + \mathbb{Q}(\mathbb{A} + \overline{\mathbb{A}}). \tag{2.39}$$

The goal of this section is to identify $\mathbb{A}$ and $\overline{\mathbb{A}}$ explicitly. Diving right in, we can write

$$\begin{aligned}
\mathbb{S}^{\mathrm{t}} &= \int_{\mathbb{X}} d^4x \left[\sqrt{g}\mathsf{C}^{\mathrm{t}}_{-} + \mathbb{Q}(\sqrt{g}\Lambda^{\mathrm{t}})\right] \\
&= \mathbb{Q}(\overline{\mathbb{V}} + \mathbb{V}) + \mathsf{C} + \mathbb{Q}\left(\int_{\mathbb{X}} d^4x\sqrt{g}\Lambda^{\mathrm{t}} - \overline{\mathbb{V}}\right) + \left(\int_{\mathbb{X}} d^4x\sqrt{g}\mathsf{C}^{\mathrm{t}}_{-} - \mathbb{Q}\mathbb{V} - \mathsf{C}\right) \\
&= \mathbb{S} + \mathbb{Q}\left(\int_{\mathbb{X}} d^4x\sqrt{g}\Lambda^{\mathrm{t}} - \overline{\mathbb{V}}\right) + \left(\int_{\mathbb{X}} d^4x\sqrt{g}\mathsf{C}^{\mathrm{t}}_{-} - \mathbb{Q}\mathbb{V} - \mathsf{C}\right).
\end{aligned} \tag{2.40}$$

Thus, rather technically, we have solved the problem for the chiral, or barred side of the action, but we can do better. Comparing this with $\overline{\mathbb{V}}$ in (2.8), we see that we have term by term agreement. The difference is the entirely in the choice of t fields and the supercovariance. We therefore write

$$\widehat{\overline{\mathbb{V}}}^{\mathrm{t}} = \int_{\mathbb{X}} d^4x\sqrt{g}\Lambda^{\mathrm{t}}, \tag{2.41}$$

where the hat over $\overline{\mathbb{V}}^{\mathrm{t}}$ indicates that we supercovariantize it as much as possible. Hence, we can rewrite the chiral difference as

$$\mathbb{Q}\left(\int_{\mathbb{X}} d^4x\sqrt{g}\Lambda^{\mathrm{t}} - \overline{\mathbb{V}}\right) = \mathbb{Q}(\widehat{\overline{\mathbb{V}}}^{\mathrm{t}} - \overline{\mathbb{V}}). \tag{2.42}$$

We next turn to the anti-chiral, or unbarred half of the action. To avoid a sea of indices in the rather tedious computation ahead, we will restrict ourselves to the IR constraints of a single U(1) vector multiplet with an arbitrary prepotential. We assure the reader that the general case follows suit [12].

To begin, expand the supercovariant terms in $\mathsf{C}^{\mathrm{t}}_{\mathrm{IR}-}$, giving

$$\begin{aligned}\mathsf{C}_{\mathrm{IR}-} = \frac{i}{2^4\pi}\Big[&-\frac{1}{2}\tau F^{\mathrm{t},-}_{\mu\nu}F^{\mathrm{t},\mu\nu}_{-} + \frac{1}{2}\tau D^{\mathrm{t}}_{\mu\nu}D^{\mathrm{t},\mu\nu} + 2\tau\nabla_\mu a\nabla^\mu\overline{a} - 2\tau\psi^{\mathrm{t}}_\mu\nabla_\nu\chi^{\mu\nu} + 2\tau\psi^{\mathrm{t}}_\mu\nabla^\mu\eta \\ &+\frac{\partial\tau}{\partial a}(F^{\mathrm{t},\mu\nu}_{-} - D^{\mathrm{t},\mu\nu})\psi^{\mathrm{t}}_\mu\psi^{\mathrm{t}}_\nu + \frac{1}{12}\sqrt{g}^{-1}\frac{\partial^2\tau}{\partial a^2}\epsilon^{\mu\nu\alpha\beta}\psi^{\mathrm{t}}_\mu\psi^{\mathrm{t}}_\nu\psi^{\mathrm{t}}_\alpha\psi^{\mathrm{t}}_\beta \\ &-\tau F^{\mathrm{t},\mu\nu}_{-}(\Psi^\sigma{}_{[\mu}\chi_{\nu]\sigma})^- + \frac{\partial\tau}{\partial a}(\Psi^\sigma{}_{[\mu}\chi_{\nu]\sigma})^-\psi^{\mathrm{t},\mu}\psi^{\mathrm{t},\nu} + 2\tau\Psi^{\mu\nu}(\nabla_\mu\overline{a})\psi^{\mathrm{t}}_\nu \\ &-\tau\Psi^\sigma{}_\sigma(\nabla_\mu\overline{a})\psi^{\mathrm{t},\mu} - 2\tau\overline{a}(\nabla^{[\mu}\Phi^{\nu]})^- F^{\mathrm{t},-}_{\mu\nu} - \frac{1}{2}\tau(\Psi^\sigma{}_{[\mu}\chi_{\nu]\sigma})^-(\Psi^{\rho[\mu}\chi^{\nu]}{}_\rho)^- \\ &+ 2\frac{\partial\tau}{\partial a}\overline{a}(\nabla^{[\mu}\Phi^{\nu]})^-\psi^{\mathrm{t}}_\mu\psi^{\mathrm{t}}_\nu - 2\tau\overline{a}(\nabla^{[\mu}\Phi^{\nu]})^-(\Psi^\sigma{}_{[\mu}\chi_{\nu]\sigma})^- - 2\tau\overline{a}^2[(\nabla_{[\mu}\Phi_{\nu]})^-]^2\Big]\end{aligned} \tag{2.43}$$

Next, we conduct the field redefintion to arrive at our Cartan model fields, and then collecting terms in increasing gravity degree, we find

$$\begin{aligned}\mathsf{C}^{\mathrm{t}}_{\mathrm{IR}-}\Big|_{\deg 0} = \frac{i}{2^4\pi}\Big[&-\frac{1}{2}\tau F^-_{\mu\nu}F^{\mu\nu}_- + \frac{1}{2}\tau D_{\mu\nu}D^{\mu\nu} + 2\tau\nabla_\sigma a\nabla^\sigma\overline{a} - 2\tau\psi_\mu\nabla_\nu\chi^{\mu\nu} + 2\tau\psi_\sigma\nabla^\sigma\eta \\ &+\frac{\partial\tau}{\partial a}(F^{\mu\nu}_- - D^{\mu\nu})\psi_\mu\psi_\nu + \frac{1}{12}\sqrt{g}^{-1}\frac{\partial^2\tau}{\partial a^2}\epsilon^{\mu\nu\alpha\beta}\psi_\mu\psi_\nu\psi_\alpha\psi_\beta\Big]\end{aligned} \tag{2.44}$$

$$\begin{aligned}\mathsf{C}^{\mathrm{t}}_{\mathrm{IR}-}\Big|_{\deg 1} = \frac{i}{2^4\pi}\Big[&-\tau\Psi^\sigma{}_{[\mu}F^{\mu\nu}_-\chi_{\nu]\sigma} + 2\tau\Psi^{\mu\nu}(\nabla_\mu\overline{a})\psi_\nu - \tau\Psi^\sigma{}_\sigma(\nabla_\mu\overline{a})\psi^\mu \\ &+\frac{\partial\tau}{\partial a}\Psi^\sigma{}_{[\mu}\chi_{\nu]\sigma}(\psi^\mu\psi^\nu)^-\Big]\end{aligned} \tag{2.45}$$

$$\begin{aligned}\mathsf{C}^{\mathrm{t}}_{\mathrm{IR}-}\Big|_{\deg 2} = \frac{i}{2^4\pi}\Big[&-2\tau(\Phi_{[\mu}\nabla_{\nu]}\overline{a})(F^{\mu\nu}_- + D^{\mu\nu}) + 2\tau\Phi^\sigma\chi_{\sigma\mu}\nabla_\nu\chi^{\mu\nu} - 2\tau\Phi^\sigma\chi_{\sigma\mu}\nabla^\mu\eta \\ &+2\frac{\partial\tau}{\partial a}\Phi_{[\mu}(\nabla_{\nu]}\overline{a})\psi^\mu\psi^\nu - 2\frac{\partial\tau}{\partial a}\Phi^\sigma(F^{\mu\nu}_- - D^{\mu\nu})\chi_{\sigma\mu}\psi_\nu\end{aligned}$$

$$-\frac{1}{3}\frac{\partial^2\tau}{\partial a^2}\epsilon^{\mu\nu\alpha\beta}\Phi^\sigma\chi_{\sigma\mu}\psi_\nu\psi_\alpha\psi_\beta - \frac{1}{2}\tau(\Psi^\sigma{}_{[\mu}\chi_{\nu]\sigma})^-(\Psi^{\rho[\mu}\chi^{\nu]}{}_\rho)^-\Big] \tag{2.46}$$

$$\mathsf{C}^{\mathrm{t}}_{\mathrm{IR}-}\Big|_{\deg 3} = \frac{i}{2^4\pi}\Big[-2\tau(\Phi_{[\mu}\nabla_{\nu]}\overline{a})^-(\Psi^{\sigma[\mu}\chi^{\nu]}{}_\sigma)^- - 2\tau\Psi^{\mu\nu}\Phi^\sigma(\nabla_\mu\overline{a})\chi_{\sigma\nu}$$
$$+\tau\Psi^\sigma{}_\sigma\Phi^\rho(\nabla_\nu\overline{a})\chi_\rho{}^\mu - 2\frac{\partial\tau}{\partial a}\Phi^\rho(\Psi^\sigma{}_{[\mu}\chi_{\nu]\sigma})^-\chi_\rho{}^\mu\psi^\nu\Big] \tag{2.47}$$

$$\mathsf{C}^{\mathrm{t}}_{\mathrm{IR}-}\Big|_{\deg 4} = \frac{i}{2^4\pi}\Big[\frac{\partial\tau}{\partial a}(F^{\mu\nu}_- - D^{\mu\nu})\Phi^\sigma\Phi^\rho\chi_{\sigma\mu}\chi_{\rho\nu} - 2\frac{\partial\tau}{\partial a}\Phi^\sigma\psi_\sigma\Phi_{[\mu}(\nabla_{\nu]}\overline{a})\chi^{\mu\nu}$$
$$+\frac{1}{2}\sqrt{g}^{-1}\frac{\partial^2\tau}{\partial a^2}\epsilon^{\mu\nu\alpha\beta}\Phi^\sigma\Phi^\rho\chi_{\sigma\mu}\chi_{\rho\nu}\psi_\alpha\psi_\beta\Big] \tag{2.48}$$

$$\mathsf{C}^{\mathrm{t}}_{\mathrm{IR}-}\Big|_{\deg 5} = \frac{i}{2^4\pi}\Big[\frac{\partial\tau}{\partial a}(\Psi^\sigma{}_{[\mu}\chi_{\nu]\sigma})^-\Phi^\rho\Phi^\gamma\chi_\rho{}^\mu\chi_\gamma{}^\nu\Big] \tag{2.49}$$

$$\mathsf{C}^{\mathrm{t}}_{\mathrm{IR}-}\Big|_{\deg 6} = \frac{i}{2^4\pi}\Big[-\frac{1}{3}\sqrt{g}^{-1}\frac{\partial^2\tau}{\partial a^2}\epsilon^{\mu\nu\alpha\beta}\Phi^\sigma\Phi^\rho\Phi^\gamma\chi_{\sigma\mu}\chi_{\rho\nu}\chi_{\gamma\alpha}\psi_\beta\Big] \tag{2.50}$$

$$\mathsf{C}^{\mathrm{t}}_{\mathrm{IR}-}\Big|_{\deg 8} = \frac{i}{2^4\pi}\Big[\frac{1}{12}\sqrt{g}^{-1}\frac{\partial^2\tau}{\partial a^2}\epsilon^{\mu\nu\alpha\beta}\Phi^\sigma\Phi^\rho\Phi^\gamma\Phi^\delta\chi_{\sigma\mu}\chi_{\rho\nu}\chi_{\gamma\alpha}\chi_{\delta\beta}\Big] \tag{2.51}$$

Multiplying the degree eight term above by our volume form $\sqrt{g}d^4x$, we see that

$$\frac{1}{12}\frac{\partial^2\tau}{\partial a^2}\epsilon^{\mu\nu\alpha\beta}\Phi^\sigma\Phi^\rho\Phi^\gamma\Phi^\delta\chi_{\sigma\mu}\chi_{\rho\nu}\chi_{\gamma\alpha}\chi_{\delta\beta}\,d^4x = \frac{1}{12}\iota_\Phi\left(\chi\wedge\iota_\Phi(\chi)\wedge\wedge\iota_\Phi(\chi)\wedge\iota_\Phi(\chi)\right) = 0, \tag{2.52}$$

as there is no support on $\mathbb{X}$ for a five form. Comparing gravity degree zero above in (2.44) to the unbarred part of $\mathbb{S}_{\mathrm{IR},0}$ in (2.12), we find exact agreement. Likewise, we have exact agreement in gravity degree one, as seen comparing (2.45) and (2.13). All together, the remaining difference of our action densities on the anti-chiral side is given by

$$\sqrt{g}\mathsf{C}^{\mathrm{t}}_{\mathrm{IR}-} - \mathbb{Q}\mathbb{V} - \mathsf{C} = \sqrt{g}\frac{i}{2^4\pi}\Big[2\tau\Phi^\sigma(F^+_{\sigma\rho} - D_{\sigma\rho})\nabla^\rho\overline{a} - 2\tau\Phi^\sigma\chi_{\sigma\mu}\nabla^\mu\eta$$
$$-\frac{1}{6}\sqrt{g}^{-1}\frac{\partial^2\tau}{\partial a^2}\Phi^\sigma\epsilon^{\mu\nu\alpha\beta}(2\chi_{\sigma\mu}\psi_\nu\psi_\alpha\psi_\beta + 3\chi_{\mu\nu}\psi_\sigma\psi_\alpha\psi_\beta)$$

$$
\begin{aligned}
&+\frac{1}{2}\frac{\partial\tau}{\partial a}\Phi^\sigma(-D_{\mu\nu}\psi_\sigma\chi^{\mu\nu}+4D_{\mu\nu}\psi^\mu\chi_\sigma{}^\nu)\\
&+\frac{1}{2}\frac{\partial\tau}{\partial a}\Phi^\sigma(-F^+_{\mu\nu}\psi_\sigma\chi^{\mu\nu}-4F^-_{\mu\nu}\psi^\mu\chi_\sigma{}^\nu-4F_{\sigma\mu}\psi_\nu\chi^{\mu\nu})\\
&+2\tau\Phi^\sigma\chi_{\sigma\mu}\nabla_\nu\chi^{\mu\nu}-\frac{1}{2}\tau\Phi^\sigma(\nabla_\sigma\chi_{\mu\nu})\chi^{\mu\nu}\\
&+\tau\Phi^\sigma\nabla_\mu(\chi_{\sigma\nu}\chi^{\mu\nu})+\frac{\partial\tau}{\partial a}(\nabla_\mu a)\Phi^\sigma\chi_{\sigma\nu}\chi^{\mu\nu}\Big]\\
+\sqrt{g}\frac{i}{2^4\pi}&\left[-2\tau(\Phi_{[\mu}\nabla_{\nu]}\overline{a})(\Psi^{\sigma[\mu}\chi^{\nu]}{}_\sigma)^- -2\tau\Phi_\mu(\nabla_\nu\overline{a})\Psi^{\nu\sigma}\chi^\mu{}_\sigma\right.\\
&\left.+\tau\Phi_{[\mu}(\nabla_{\nu]}\overline{a})\Psi^\sigma{}_\sigma\chi^{\mu\nu}-2\frac{\partial\tau}{\partial a}\Phi^\rho(\Psi^\sigma{}_{[\mu}\chi_{\nu]\sigma})^-\chi_\rho{}^\mu\psi^\nu\right]\\
+\sqrt{g}\frac{i}{2^4\pi}&\left[\frac{\partial\tau}{\partial a}\Phi^\sigma\Phi^\rho(F_-^{\mu\nu}-D^{\mu\nu})\chi_{\sigma\mu}\chi_{\rho\nu}-2\frac{\partial\tau}{\partial a}\Phi^\sigma\Phi^\rho\psi_\sigma(\nabla^\mu\overline{a})\chi_{\rho\mu}\right.\\
&\left.+\frac{1}{2}\sqrt{g}^{-1}\frac{\partial^2\tau}{\partial a^2}\Phi^\rho\Phi^\sigma\epsilon^{\mu\nu\alpha\beta}\chi_{\rho\mu}\chi_{\sigma\nu}\psi_\alpha\psi_\beta\right]\\
+\sqrt{g}\frac{i}{2^4\pi}&\left[\frac{\partial\tau}{\partial a}(\Psi^\sigma{}_{[\mu}\chi_{\nu]\sigma})^-\Phi^\rho\Phi^\gamma\chi_\rho{}^\mu\chi_\gamma{}^\nu\right]\\
+\frac{i}{2^4\pi}&\left[-\frac{1}{3}\frac{\partial^2\tau}{\partial a^2}\Phi^\rho\Phi^\sigma\Phi^\gamma\epsilon^{\mu\nu\alpha\beta}\chi_{\rho\mu}\chi_{\sigma\nu}\chi_{\gamma\alpha}\psi_\beta\right],
\end{aligned}
\tag{2.53}
$$

where, as usual, we have split the terms in the increasing gravity degree.

Let's begin by settling degree two, note that

$$
-\frac{1}{6}d^4x\frac{\partial^2\tau}{\partial a^2}\Phi^\sigma\epsilon^{\mu\nu\alpha\beta}(2\chi_{\sigma\mu}\psi_\nu\psi_\alpha\psi_\beta+3\chi_{\mu\nu}\psi_\sigma\psi_\alpha\psi_\beta)=-\frac{1}{6}\iota_\Phi(\chi\wedge\psi\wedge\psi\wedge\psi)=0
\tag{2.54}
$$

Further, we have

$$
\tau\Phi^\sigma\nabla_\mu(\chi_{\sigma\nu}\chi^{\mu\nu})=\tau\Phi^\sigma(\nabla_\mu\chi_{\sigma\nu})\chi^{\mu\nu}-\tau\Phi^\sigma\chi_{\sigma\mu}\nabla_\nu\chi^{\mu\nu}
\tag{2.55}
$$

as well as

$$
-\frac{1}{2}\tau\sqrt{g}\Phi^\sigma(\nabla_\sigma\chi_{\mu\nu})\chi^{\mu\nu}=-\frac{1}{4}\tau\epsilon^{\mu\nu\alpha\beta}\Phi^\sigma(\nabla_\sigma\chi_{\mu\nu})\chi_{\alpha\beta}
$$

$$= -\frac{1}{2}\tau\epsilon^{\mu\nu\alpha\beta}\Phi^\sigma(\nabla_\mu\chi_{\sigma\nu})\chi_{\alpha\beta} - \frac{1}{2}\tau\epsilon^{\mu\nu\alpha\beta}\Phi^\sigma(\nabla_\mu\chi_{\nu\alpha})\chi_{\sigma\beta}$$
$$= -\sqrt{g}\tau\Phi^\sigma(\nabla_\mu\chi_{\sigma\nu})\chi^{\mu\nu} - \sqrt{g}\tau\Phi^\sigma\chi_{\sigma\mu}(\nabla_\nu\chi^{\mu\nu}). \tag{2.56}$$

Turning to the $\Phi F_A\psi\chi$ terms, note that

$$\begin{aligned}
-2\sqrt{g}\frac{\partial\tau}{\partial a}\Phi^\sigma F_{\sigma\mu}\psi_\nu\chi^{\mu\nu} &= -\frac{\partial\tau}{\partial a}\epsilon^{\mu\nu\alpha\beta}\Phi^\sigma F_{\sigma\mu}\psi_\nu\chi_{\alpha\beta} \\
&= \frac{1}{2}\frac{\partial\tau}{\partial a}\epsilon^{\mu\nu\alpha\beta}\Phi^\sigma F_{\mu\nu}\psi_\sigma\chi_{\alpha\beta} - \frac{\partial\tau}{\partial a}\epsilon^{\mu\nu\alpha\beta}\Phi^\sigma F_{\mu\nu}\psi_\alpha\chi_{\sigma\beta} \\
&= \sqrt{g}\frac{\partial\tau}{\partial a}\Phi^\sigma F^+_{\mu\nu}\psi_\sigma\chi^{\mu\nu} - \frac{\partial\tau}{\partial a}\epsilon^{\mu\nu\alpha\beta}\Phi^\sigma F_{\mu\nu}\psi_\alpha\chi_{\sigma\beta},
\end{aligned} \tag{2.57}$$

and

$$-2\sqrt{g}\frac{\partial\tau}{\partial a}\Phi^\sigma F^-_{\mu\nu}\psi^\mu\chi_\sigma{}^\nu = -\sqrt{g}\frac{\partial\tau}{\partial a}\Phi^\sigma F_{\mu\nu}\psi^\mu\chi_\sigma{}^\nu + \frac{1}{2}\frac{\partial\tau}{\partial a}\epsilon^{\mu\nu\alpha\beta}\Phi^\sigma F_{\mu\nu}\psi_\alpha\chi_{\sigma\beta}. \tag{2.58}$$

This gives

$$\frac{1}{2}\sqrt{g}\frac{\partial\tau}{\partial a}\Phi^\sigma(-F^+_{\mu\nu}\psi_\sigma\chi^{\mu\nu} - 4F^-_{\mu\nu}\psi^\mu\chi_\sigma{}^\nu - 4F_{\sigma\mu}\psi_\nu\chi^{\mu\nu}) = \frac{1}{2}\sqrt{g}\frac{\partial\tau}{\partial a}\Phi^\sigma\left[F^+_{\mu\nu}\psi_\sigma\chi^{\mu\nu} - 4F^+_{\mu\nu}\psi^\nu\chi_\sigma{}^\nu\right]. \tag{2.59}$$

Thus, we can clean up the remaining terms into

$$\begin{aligned}
\sqrt{g}\mathsf{C}^{\mathrm{t}}_{\mathrm{IR}-} - \mathbb{QV} - \mathsf{C} = \sqrt{g}\frac{i}{2^4\pi}\Big[&2\tau\Phi^\sigma(F^+_{\sigma\rho} - D_{\sigma\rho})\nabla^\rho\overline{a} - 2\tau\Phi^\sigma\chi_{\sigma\mu}\nabla^\mu\eta \\
&+\frac{1}{2}\frac{\partial\tau}{\partial a}\Phi^\sigma(-D_{\mu\nu}\psi_\sigma\chi^{\mu\nu} + 4D_{\mu\nu}\psi^\mu\chi_\sigma{}^\nu) \\
&+\frac{1}{2}\frac{\partial\tau}{\partial a}\Phi^\sigma(F^+_{\mu\nu}\psi_\sigma\chi^{\mu\nu} - 4F^+_{\mu\nu}\psi^\nu\chi_\sigma{}^\nu) \\
&+\frac{\partial\tau}{\partial a}(\nabla_\mu a)\Phi^\sigma\chi_{\sigma\nu}\chi^{\mu\nu}\Big] \\
+ \sqrt{g}\frac{i}{2^4\pi}\Big[&-2\tau(\Phi_{[\mu}\nabla_{\nu]}\overline{a})(\Psi^{\sigma[\mu}\chi^{\nu]}{}_\sigma)^- - 2\tau\Phi_\mu(\nabla_\nu\overline{a})\Psi^{\nu\sigma}\chi^\mu{}_\sigma
\end{aligned}$$

$$
\begin{aligned}
&+\tau\Phi_{[\mu}(\nabla_{\nu]}\overline{a})\Psi^{\sigma}{}_{\sigma}\chi^{\mu\nu} - 2\frac{\partial\tau}{\partial a}\Phi^{\rho}(\Psi^{\sigma}{}_{[\mu}\chi_{\nu]\sigma})^{-}\chi_{\rho}{}^{\mu}\psi^{\nu}\Bigg] \\
&+\sqrt{g}\frac{i}{2^4\pi}\left[\frac{\partial\tau}{\partial a}\Phi^{\sigma}\Phi^{\rho}(F_{-}^{\mu\nu} - D^{\mu\nu})\chi_{\sigma\mu}\chi_{\rho\nu} - 2\frac{\partial\tau}{\partial a}\Phi^{\sigma}\Phi^{\rho}\psi_{\sigma}(\nabla^{\mu}\overline{a})\chi_{\rho\mu}\right. \\
&\qquad\left. +\frac{1}{2}\sqrt{g}^{-1}\frac{\partial^2\tau}{\partial a^2}\Phi^{\rho}\Phi^{\sigma}\epsilon^{\mu\nu\alpha\beta}\chi_{\rho\mu}\chi_{\sigma\nu}\psi_{\alpha}\psi_{\beta}\right] \\
&+\sqrt{g}\frac{i}{2^4\pi}\left[\frac{\partial\tau}{\partial a}(\Psi^{\sigma}{}_{[\mu}\chi_{\nu]\sigma})^{-}\Phi^{\rho}\Phi^{\gamma}\chi_{\rho}{}^{\mu}\chi_{\gamma}{}^{\nu}\right] \\
&+\frac{i}{2^4\pi}\left[-\frac{1}{3}\frac{\partial^2\tau}{\partial a^2}\Phi^{\rho}\Phi^{\sigma}\Phi^{\gamma}\epsilon^{\mu\nu\alpha\beta}\chi_{\rho\mu}\chi_{\sigma\nu}\chi_{\gamma\alpha}\psi_{\beta}\right] \qquad (2.60)
\end{aligned}
$$

We will now show that the above is an entirely $\mathbb{Q}$-exact expression. Starting with the highest degree, we have

$$
\begin{aligned}
\mathbb{Q}\left(\frac{1}{6}\frac{\partial\tau}{\partial a}\Phi^{\rho}\Phi^{\sigma}\epsilon^{\mu\nu\alpha\beta}\chi_{\mu\nu}\chi_{\rho\alpha}\chi_{\sigma\beta}\right) &= -\frac{1}{6}\frac{\partial^2\tau}{\partial a^2}\Phi^{\sigma}\psi_{\sigma}\Phi^{\rho}\Phi^{\gamma}\epsilon^{\mu\nu\alpha\beta}\chi_{\mu\nu}\chi_{\rho\alpha}\chi_{\gamma\beta} \\
&\quad - \frac{1}{6}\frac{\partial\tau}{\partial a}\Phi^{\rho}\Phi^{\sigma}\epsilon^{\mu\nu\alpha\beta}(D_{\mu\nu} - F^{+}_{\mu\nu} + (\Psi^{\gamma}{}_{[\mu}\chi_{\nu]\gamma})^{-})\chi_{\rho\alpha}\chi_{\sigma\beta} \\
&\quad - \frac{1}{3}\frac{\partial\tau}{\partial a}\Phi^{\rho}\Phi^{\sigma}\epsilon^{\mu\nu\alpha\beta}\chi_{\mu\nu}(D_{\rho\alpha} - F^{+}_{\rho\alpha} + (\Psi^{\gamma}{}_{[\rho}\chi_{\alpha]\gamma})^{-})\chi_{\sigma\beta} \\
&= -\frac{1}{3}\frac{\partial^2\tau}{\partial a^2}\Phi^{\rho}\Phi^{\sigma}\Phi^{\gamma}\epsilon^{\mu\nu\alpha\beta}\chi_{\rho\mu}\chi_{\sigma\nu}\chi_{\gamma\alpha}\psi_{\beta} \\
&\quad + \sqrt{g}\frac{\partial\tau}{\partial a}\Phi^{\rho}\Phi^{\sigma}(\Psi^{\gamma}{}_{[\mu}\chi_{\nu]\gamma})^{-}\chi_{\rho}{}^{\mu}\chi_{\sigma}{}^{\nu} \\
&\quad + \sqrt{g}\frac{\partial\tau}{\partial a}\Phi^{\sigma}\Phi^{\rho}(F_{+}^{\mu\nu} - D^{\mu\nu})\chi_{\sigma\mu}\chi_{\rho\nu}. \qquad (2.61)
\end{aligned}
$$

Above, we have used the fact that

$$
\frac{1}{3}\Phi^{\sigma}(\epsilon^{\mu\nu\alpha\beta}\chi_{\sigma\mu}\Phi^{\rho}\chi_{\rho\nu}\Phi^{\gamma}\chi_{\gamma\alpha}\psi_{\beta}) = \frac{1}{6}\Phi^{\sigma}\psi_{\sigma}\epsilon^{\mu\nu\alpha\beta}\chi_{\mu\nu}\Phi^{\rho}\chi_{\rho\alpha}\Phi^{\gamma}\chi_{\gamma\beta}, \qquad (2.62)
$$

which follows from another “unsupported five-form" argument, that is, we have

$$
0 = \iota_{\Phi}(\chi \wedge \iota_{\Phi}(\chi) \wedge \iota_{\Phi}(\chi) \wedge \psi)
$$

$$\sim \int_{\mathbb{X}} d^4x \left[\frac{1}{3}\Phi^\sigma(\epsilon^{\mu\nu\alpha\beta}\chi_{\sigma\mu}\Phi^\rho\chi_{\rho\nu}\Phi^\gamma\chi_{\gamma\alpha}\psi_\beta) + \frac{1}{6}\Phi^\sigma(\epsilon^{\mu\nu\alpha\beta}\chi_{\mu\nu}\Phi^\rho\chi_{\rho\alpha}\Phi^\gamma\chi_{\gamma\beta}\psi_\sigma)\right]. \quad (2.63)$$

Moving down to the next highest remaining degree, consider

$$\begin{aligned}\mathbb{Q}\left(\frac{1}{2}\frac{\partial\tau}{\partial a}\Phi^\sigma\epsilon^{\mu\nu\alpha\beta}\chi_{\sigma\mu}\chi_{\alpha\beta}\psi_\nu\right) = &-\frac{1}{2}\frac{\partial^2\tau}{\partial a^2}\Phi^\rho\psi_\rho\Phi^\sigma\epsilon^{\mu\nu\alpha\beta}\chi_{\sigma\mu}\chi_{\alpha\beta}\psi_\nu \\ &+\frac{1}{2}\frac{\partial\tau}{\partial a}\Phi^\sigma\epsilon^{\mu\nu\alpha\beta}(F^+_{\sigma\mu} - D_{\sigma\mu})\chi_{\alpha\beta}\psi_\nu \\ &-\frac{1}{2}\frac{\partial\tau}{\partial a}\Phi^\sigma\epsilon^{\mu\nu\alpha\beta}\chi_{\sigma\mu}(F^+_{\alpha\beta} - D_{\alpha\beta})\psi_\nu \\ &+\frac{1}{2}\frac{\partial\tau}{\partial a}\Phi^\sigma\epsilon^{\mu\nu\alpha\beta}(\Psi^\rho{}_{[\sigma}\chi_{\mu]\rho})^-\chi_{\alpha\beta}\psi_\nu \\ &-\frac{1}{2}\frac{\partial\tau}{\partial a}\Phi^\sigma\epsilon^{\mu\nu\alpha\beta}\chi_{\sigma\mu}(\Psi^\rho{}_{[\alpha}\chi_{\beta]\rho})^-\psi_\nu \\ &-\frac{1}{2}\frac{\partial\tau}{\partial a}\Phi^\sigma\epsilon^{\mu\nu\alpha\beta}\chi_{\sigma\mu}\chi_{\alpha\beta}\nabla_\nu a \\ &+\frac{1}{2}\frac{\partial\tau}{\partial a}\Phi^\sigma\epsilon^{\mu\nu\alpha\beta}\chi_{\sigma\mu}\chi_{\alpha\beta}\Phi^\rho F_{\rho\nu}. \end{aligned} \quad (2.64)$$

Note that

$$\begin{aligned}\frac{1}{2}\Phi^\sigma\epsilon^{\mu\nu\alpha\beta}(F^+_{\sigma\mu}\chi_{\alpha\beta}\psi_\nu - \chi_{\sigma\mu}F^+_{\alpha\beta}\psi_\nu) &= -\Phi^\sigma\epsilon^{\mu\nu\alpha\beta}(\chi_{\sigma\mu}F^+_{\alpha\beta}\psi_\nu - \frac{1}{4}F^+_{\mu\nu}\chi_{\alpha\beta}\psi_\sigma) \\ &= -2\sqrt{g}\Phi^\sigma F^+_{\mu\nu}\psi^\mu\chi_\sigma{}^\nu + \frac{1}{2}\sqrt{g}\Phi^\sigma F^+_{\mu\nu}\psi_\sigma\chi^{\mu\nu}. \end{aligned} \quad (2.65)$$

Likewise, analogous arguments show that

$$-\frac{1}{2}\Phi^\sigma\epsilon^{\mu\nu\alpha\beta}(D_{\sigma\mu}\chi_{\alpha\beta}\psi_\nu + \chi_{\sigma\mu}D_{\alpha\beta}\psi_\nu) = 2\sqrt{g}\Phi^\sigma D_{\mu\nu}\psi^\mu\chi_\sigma{}^\nu - \frac{1}{2}\sqrt{g}\Phi^\sigma D_{\mu\nu}\psi_\sigma\chi^{\mu\nu}, \quad (2.66)$$

and

$$-\frac{1}{2}\Phi^{\sigma}\epsilon^{\mu\nu\alpha\beta}((\Psi^{\rho}{}_{[\sigma}\chi_{\mu]\rho})^{-}\chi_{\alpha\beta}\psi_{\nu}+\chi_{\sigma\mu}(\Psi^{\rho}{}_{[\alpha}\chi_{\beta]\rho})^{-}\psi_{\nu})=-2\sqrt{g}\Phi^{\rho}(\Psi^{\sigma}{}_{[\mu}\chi_{\nu]\sigma})^{-}\chi_{\rho}{}^{\mu}\psi^{\nu}. \tag{2.67}$$

Here, in the final case, the potential second term on the right hand side vanishes due to the contraction of the anti-self-dual form $(\Psi\chi)^{-}$ with the self-dual χ. With these, together with another bout of index shifting, we compute

$$\begin{aligned}\mathbb{Q}\left(\frac{1}{2}\frac{\partial\tau}{\partial a}\Phi^{\sigma}\epsilon^{\mu\nu\alpha\beta}\chi_{\sigma\mu}\chi_{\alpha\beta}\psi_{\nu}\right) &= \sqrt{g}\frac{\partial\tau}{\partial a}(\nabla_{\mu}a)\Phi^{\sigma}\chi_{\sigma\nu}\chi^{\mu\nu}\\ &\quad+\frac{1}{2}\frac{\partial\tau}{\partial a}\Phi^{\sigma}(-D_{\mu\nu}\psi_{\sigma}\chi^{\mu\nu}+4D_{\mu\nu}\psi^{\mu}\chi_{\sigma}{}^{\nu})\\ &\quad+\frac{1}{2}\frac{\partial\tau}{\partial a}\Phi^{\sigma}(F^{+}_{\mu\nu}\psi_{\sigma}\chi^{\mu\nu}-4F^{+}_{\mu\nu}\psi^{\nu}\chi_{\sigma}{}^{\nu})\\ &\quad-2\sqrt{g}\frac{\partial\tau}{\partial a}\Phi^{\rho}(\Psi^{\sigma}{}_{[\mu}\chi_{\nu]\sigma})^{-}\chi_{\rho}{}^{\mu}\psi^{\nu}\\ &\quad-\frac{1}{2}\frac{\partial\tau}{\partial a}\Phi^{\sigma}\Phi^{\rho}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}\chi_{\sigma\alpha}\chi_{\rho\beta}\\ &\quad+\frac{1}{2}\frac{\partial^{2}\tau}{\partial a^{2}}\Phi^{\rho}\Phi^{\sigma}\epsilon^{\mu\nu\alpha\beta}\chi_{\rho\mu}\chi_{\sigma\nu}\psi_{\alpha}\psi_{\beta}\end{aligned} \tag{2.68}$$

Here, the term in the penultimate line is precisely that required to combine with the final term in (2.61) to give the desired F_{A}^{-} dependence found in the degree four term of our difference (2.60).

Finally, consider

$$\begin{aligned}\mathbb{Q}\left(\tau\Phi_{\mu}\epsilon^{\mu\nu\alpha\beta}\chi_{\alpha\beta}\nabla_{\nu}\bar{a}\right) &= -\frac{\partial\tau}{\partial a}\Phi^{\sigma}\Phi_{\mu}\epsilon^{\mu\nu\alpha\beta}\psi_{\sigma}\chi_{\alpha\beta}\nabla_{\nu}\bar{a}+\tau\Phi^{\sigma}\Psi_{\sigma\mu}\epsilon^{\mu\nu\alpha\beta}\chi_{\alpha\beta}\nabla_{\nu}\bar{a}\\ &\quad+\tau\Phi_{\mu}\epsilon^{\mu\nu\alpha\beta}(F^{+}_{\alpha\beta}-D_{\alpha\beta}-(\Psi^{\sigma}{}_{[\alpha}\chi_{\beta]\sigma})^{-})\nabla_{\nu}\bar{a}-\tau\Phi_{\mu}\epsilon^{\mu\nu\alpha\beta}\chi_{\alpha\beta}\nabla_{\nu}\eta.\\ &= 2\sqrt{g}\tau\Phi^{\sigma}(F^{+}_{\sigma\rho}-D_{\sigma\rho})\nabla^{\rho}\bar{a}-2\sqrt{g}\tau\Phi^{\sigma}\chi_{\sigma\mu}\nabla^{\mu}\eta\\ &\quad-2\sqrt{g}\tau\Phi_{\mu}\nabla_{\nu}\bar{a}(\Psi^{\mu\sigma}\chi^{\nu}{}_{\sigma})+2\sqrt{g}\tau\Phi_{\mu}\nabla_{\nu}\bar{a}(\Psi^{\sigma[\mu}\chi^{\nu]}{}_{\sigma})^{-}\end{aligned}$$

$$- 2\sqrt{g}\frac{\partial\tau}{\partial a}\Phi^{\sigma}\Phi^{\rho}\psi_{\sigma}(\nabla^{\mu}\overline{a})\chi_{\rho\mu} \tag{2.69}$$

which, after careful inspection, accounts for all the remaining terms in (2.60).

Therefore, all together, define $\mathbb{A}$ and $\overline{\mathbb{A}}$ as

$$\mathbb{A} = \frac{i}{2^4\pi}\int_{\mathbb{X}} d^4x\sqrt{g}\left[2\tau\Phi^{\rho}\chi_{\rho\mu}\nabla^{\mu}\overline{a} + \frac{\partial\tau}{\partial a}\Phi^{\rho}\chi_{\rho\mu}\chi^{\mu\nu}\psi_{\nu} + \frac{1}{3}\frac{\partial\tau}{\partial a}\Phi^{\rho}\Phi^{\sigma}\chi_{\rho\mu}\chi_{\sigma\nu}\chi^{\mu\nu}\right], \tag{2.70}$$

$$\overline{\mathbb{A}} = \widehat{\overline{\mathbb{V}}}^{\mathrm{t}} - \overline{\mathbb{V}}. \tag{2.71}$$

There is no a prior reason to favor $\mathbb{S}$ over $\mathbb{S}^{\mathrm{t}}$, but given the simplicity of the transformations of the Cartan model fields and the fact the $\mathbb{S}$ terminates at gravity degree two, we will move forward with $\mathbb{S}$. Nevertheless, in Appendix , we use our knowledge of the form of $\mathbb{S}$ to write the anti-chiral part of $\mathbb{S}^{\mathrm{t}}$ as a maximal exact and non-exact splitting.

With the action behind us, the hero of our hero enters the scene...

3 The Invariants

3.1 Overview

Our ultimate goal is to build equivariant classes for $H_{\mathsf{Diff}_+(\mathbb{X})}(\mathsf{Met}(\mathbb{X}))$ which generalize the Donaldson-Witten partition function,

$$\mathsf{Z}_{\mathrm{DW}}[g] = \int [d\mathrm{VM}]e^{-S_{\mathrm{UV}}}. \tag{3.1}$$

Indeed, we wish to view $\mathsf{Z}_{\mathrm{DW}}[g]$ as an element of $H^0_{\mathsf{Diff}_+(\mathbb{X})}(\mathsf{Met}(\mathbb{X}))$. Our new invariants will then be higher degree elements of this equivariant cohomology, which can be formally understood as differential forms on $\mathsf{Met}(\mathbb{X})$ which are invariant under the action of $\mathsf{Diff}_+(\mathbb{X})$.

Therefore, and perhaps unsurprising, we take our new invariant $\mathsf{Z}[g, \Psi, \Phi]$ to be (3.1) with the action S_{UV} replaced by $\mathbb{S}_{\text{UV}}$, the $\mathbb{Q}$-closed action under the conditions of (2.21). We will see how, in much the same sense that the original Donaldson-Witten invariants were a generating function for the polynomial invariants, a simple chain map argument reveals that $\mathsf{Z}[g, \Psi, \Phi]$ generates an infinite tower of higher degree elements of $H_{\mathsf{Diff}_+(\mathbb{X})}(\mathsf{Met}(\mathbb{X}))$.

In this section we will present the salient features of $\mathsf{Z}[g, \Psi, \Phi]$, and then, after getting our hands dirty with some interrelations between the different gravity degree parts of our action, we will conduct an preliminary exploration of both $\mathsf{Z}^{[0]}$ and $\mathsf{Z}^{[1]}$.

3.2 The Partition Function

Consider the UV limit of the action constructed in Section 2.2. Recall that

$$\mathbb{Q}\mathbb{S}_{\text{UV}} = 0. \tag{3.2}$$

Thus, when we exponentiate the action, we have $\mathbb{Q}$-closure as

$$\mathbb{Q}\left(e^{-S_{\text{UV}}}\right) = (-\mathbb{Q}\mathbb{S}_{\text{UV}})e^{-\mathbb{S}_{\text{UV}}} = 0. \tag{3.3}$$

In order to obtain our desired diffeomorphism invariants, we need to integrate out the vector multiplet fields. Thus, we have our *Family Donaldson-Witten partition function* as

$$\mathsf{Z}[g, \Psi, \Phi] = \int [d\text{VM}]\, e^{-\mathbb{S}^{\text{UV}}} \tag{3.4}$$

Note that we have here the full $\mathbb{S}_{\rm UV}$, not $S_{\rm UV}$.

The role of integrating over the vector multiplet requires some understanding. Our total complex is given by

$$\begin{aligned}\widetilde{\mathbb{E}}_g = S^\bullet(\mathrm{Lie}\mathbb{G}) \otimes \Omega^\bullet(\mathbb{M}) \otimes \Omega^{2,+}_g(\mathbb{X}, \mathrm{ad}P) \otimes \Pi\Omega^{2,+}_g(\mathbb{X}, \mathrm{ad}P) \\ \otimes \Omega^0(\mathbb{X}, \mathrm{ad}P) \otimes \Pi\Omega^0(\mathbb{X}, \mathrm{ad}P)\end{aligned} \tag{3.5}$$

Recall that the first two tensorands are the base of our Cartan model and are generated by $(g, A, \psi, \Psi, \phi, \Phi)$. The next pair are the localization multiplet module $(H/D, \chi)$, and the final pair are the projection multiplet module (λ, η). We also mention that, due to the dependence of self-duality on a metric (or more accurately, on the conformal class of a metric), the entire complex depends on a fixed choice of metric. Then, following the tutelage of the Cartan model, the subcomplex on which $\mathbb{Q}^2 = 0$ is then given by

$$\mathbb{E}_g = (\widetilde{\mathbb{E}}_g)^{\mathbb{G}}, \tag{3.6}$$

where, as before, the raised $\mathbb{G}$ denotes the fact that we are restricting to the $\mathbb{G}$-invariant subcomplex. The base space, over which we understand $\mathbb{E}_g$ as a bundle, is given by the complex

$$\mathbb{B} = (S^\bullet(\mathsf{LieDiff}_+(\mathbb{X})) \otimes \Omega^\bullet(\mathsf{Met}(\mathbb{X})))^{\mathsf{Diff}_+(\mathbb{X})} \tag{3.7}$$

The differential for this complex is given by d of (1.10)-(1.11).

Projecting down from $\mathbb{E}_g$ to $\mathbb{B}$ defines a chain map

$$\pi_* : \mathbb{E}_g \longrightarrow \mathbb{B} \tag{3.8}$$

As such, the following diagram commutes.

$$\begin{array}{ccc} \mathbb{E}_g & \xrightarrow{\mathbb{Q}} & \mathbb{E}_g \\ {\scriptstyle \pi_*}\downarrow & & \downarrow{\scriptstyle \pi_*} \\ \mathbb{B} & \xrightarrow{\mathsf{d}} & \mathbb{B} \end{array}$$

At a formal level, the map π_* is equivalent to the path integral over the twisted vector multiplet. Therefore, the above diagram can be expressed as

$$\int [d\mathrm{VM}]\mathbb{Q}A = \mathsf{d}\left(\int [d\mathrm{VM}]A\right), \tag{3.9}$$

for all $A \in \mathbb{E}_g$. Hence, we conclude that

$$\mathsf{dZ}[g,\Psi,\Phi] = \int [d\mathrm{VM}]\mathbb{Q}\left(e^{-\mathbb{S}_{\mathrm{UV}}}\right) = 0. \tag{3.10}$$

Since $\exp[-\mathbb{S}_{\mathrm{UV}}]$ cannot be written as a $\mathbb{Q}$-exact element of $\mathbb{E}_g$, we then conclude that $\mathsf{Z}[g,\Psi,\Phi]$ is a nontrivial element of $H_{\mathsf{Diff}_+(\mathbb{X})}(\mathsf{Met}(\mathbb{X}))$.

Now, before expanding $\mathsf{Z}[g,\Psi,\Phi]$ into elements of homogenous gravity degree, we must investigate some properties of the action $\mathbb{S}_{\mathrm{UV}}$.

3.3 Preliminary Relations

Our action $\mathbb{S}_{\mathrm{UV}}$ has the explicit form

$$\mathbb{S}_{\mathrm{UV}} = S_{\mathrm{UV}} - \frac{1}{2}\int_{\mathbb{X}} d^4x\sqrt{g}\Psi^{\mu\nu}\Lambda^{\mathrm{UV}}_{\mu\nu} + \int_{\mathbb{X}} d^4x\sqrt{g}\Phi^{\sigma}Z^{\mathrm{UV}}_{\sigma} + \int_{\mathbb{X}} d^4x\sqrt{g}\Psi^{\mu\sigma}\Psi^{\nu}{}_{\sigma}\Upsilon^{\mathrm{UV}}_{\mu\nu}, \tag{3.11}$$

where we have collected terms into increasing gravity degree, so that $S_{\mathrm{UV}}, \Lambda^{\mathrm{UV}}, Z^{\mathrm{UV}}, \Upsilon^{\mathrm{UV}}$ are each functionals of the twisted vector multiplet fields and metric only. Since the entire action must have total degree zero, the gauge degrees of these functionals must be negative. We repeat the definition for each term. At degree zero, we have the original Donaldson-Witten action, with

$$S_{\mathrm{UV}} = S^{\mathrm{Loc}} + S^{\mathrm{Pro}} + S^{\mathrm{Pot}} - 2\pi i \tau_0 k \tag{3.12}$$

$$= \mathcal{Q}(V_{\mathrm{UV}}^{\mathrm{Loc}} + V_{\mathrm{UV}}^{\mathrm{Pro}} + V_{\mathrm{UV}}^{\mathrm{Pot}}) + \frac{i\tau_0}{8\pi^2}\int_{\mathbb{X}} \mathrm{Tr} F_A \wedge F_A. \tag{3.13}$$

In a slight change of overall normalization from (0.137)-(0.139) to agree with (2.12), we have

$$S_{\mathrm{UV}}^{\mathrm{Loc}} = \frac{1}{2g_0^2}\int_{\mathbb{X}} d^4x\sqrt{g}\mathrm{Tr}\left[\frac{1}{2}F_{\mu\nu}^{+}F_{+}^{\mu\nu} - \frac{1}{2}D_{\mu\nu}D^{\mu\nu} - 2\chi_{\mu\nu}(D^\mu\psi^\nu)_+ + \frac{1}{2}\chi_{\mu\nu}[\phi,\chi^{\mu\nu}]\right], \tag{3.14}$$

$$S_{\mathrm{UV}}^{\mathrm{Pro}} = -\frac{1}{2g_0^2}\int_{\mathbb{X}} d^4x\sqrt{g}\mathrm{Tr}\left[2\eta D_\mu\psi^\mu + 2\lambda[\psi_\mu,\psi^\mu] - 2\lambda D_\mu D^\mu\phi\right], \tag{3.15}$$

$$S_{\mathrm{UV}}^{\mathrm{Pot}} = -\frac{1}{2g_0^2}\int_{\mathbb{X}} d^4x\sqrt{g}\mathrm{Tr}\left[2[\phi,\lambda]^2 - 2\eta[\phi,\eta]\right], \tag{3.16}$$

$$V_{\mathrm{UV}}^{\mathrm{Loc}} = \frac{1}{2g_0^2}\int_{\mathbb{X}} d^4x\sqrt{g}\mathrm{Tr}\left[\frac{1}{2}(F_{\mu\nu}^{+} + D_{\mu\nu})\chi^{\mu\nu}\right], \tag{3.17}$$

$$V_{\mathrm{UV}}^{\mathrm{Pro}} = -\frac{1}{2g_0^2}\int_{\mathbb{X}} d^4x\sqrt{g}\mathrm{Tr}\left[2\lambda D_\mu\psi^\mu\right], \tag{3.18}$$

$$V_{\mathrm{UV}}^{\mathrm{Pot}} = -\frac{1}{2g_0^2}\int_{\mathbb{X}} d^4x\sqrt{g}\mathrm{Tr}\left[2\eta[\phi,\lambda]\right]. \tag{3.19}$$

At degree one, we have

$$\Lambda_{\mu\nu}^{\mathrm{UV}} = \frac{1}{2g_0^2}\mathrm{Tr}\left[\frac{1}{2}F_\mu{}^\rho\chi_{\nu\rho} + \frac{1}{2}F_\nu{}^\rho\chi_{\mu\rho} - \frac{1}{4}g_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}\right.$$
$$\left. + 2(D_\mu\lambda)\psi_\nu + 2(D_\nu\lambda)\psi_\nu - 2g_{\mu\nu}(D_\sigma\lambda)\psi^\sigma + 2g_{\mu\nu}\eta[\phi,\lambda]\right]$$

Here, we recall that $\Lambda^{\rm UV}_{\mu\nu}$ satisfies the relation

$$T^{\rm UV}_{\mu\nu} = \mathcal{Q}\Lambda^{\rm UV}_{\mu\nu}, \tag{3.20}$$

where $T^{\rm UV}_{\mu\nu}$ is the energy-momentum tensor of the original Dondaldson-Witten theory, and thus, in our new language, satisfies[41],

$$\widetilde{\mathsf{d}}S_{\rm UV} = \frac{1}{2}\int_{\mathbb{X}} d^4x\sqrt{g}\Psi^{\mu\nu}T^{\rm UV}_{\mu\nu}. \tag{3.21}$$

At degree two we have two terms. The first is

$$Z_\sigma = \frac{1}{2g_0^2}\mathrm{Tr}\left[\frac{1}{4}\chi^{\mu\nu}D_\sigma\chi_{\mu\nu} - \frac{1}{2}D_\mu(\chi^{\mu\nu}\chi_{\sigma\nu}) + 2(D_\rho\lambda)F_\sigma{}^\rho - 2(D_\sigma\lambda)[\phi,\lambda] - 2\eta[\psi_\sigma,\lambda]\right]. \tag{3.22}$$

The second degree two term has the form

$$\Upsilon_{\mu\nu} = \frac{1}{8g_0^2}\mathrm{Tr}\left[\chi_\mu{}^\sigma\chi_{\nu\sigma}\right]. \tag{3.23}$$

This action was constructed by taking, as definition,

$$\mathbb{S}_{\rm UV} = \mathbb{Q}(V^{\rm Loc}_{\rm UV} + V^{\rm Pro}_{\rm UV} + V^{\rm Pot}_{\rm UV}) + \frac{i\tau_0}{8\pi^2}\int_{\mathbb{X}}\mathrm{Tr}F_A\wedge F_A. \tag{3.24}$$

In order to streamline future computations, let us understand how the action closes under $\mathbb{Q}$. We can split the action into bidegree and write

$$\mathbb{S}_{\rm UV} = \mathbb{S}^{(0,0)}_{\rm UV} + \mathbb{S}^{(-1,1)}_{\rm UV} + \mathbb{S}^{(-2,2)}_{\rm UV}. \tag{3.25}$$

[41]It is not the energy-momentum tensor of the full theory of twisted supergravity

With this, we can understand the action of $\mathbb{Q}$ in bidegrees as relations between terms of the same bidegree. First, we have

$$\mathbb{Q}^{(1,0)}\mathbb{S}_{\mathrm{UV}}^{(0,0)} = \mathcal{Q}S_{\mathrm{UV}} = 0, \tag{3.26}$$

which is just the statement that $\mathcal{Q}^2 = \delta_\phi$ and that S_{UV} is gauge invariant. Next, we have

$$\mathbb{Q}^{(0,1)}\mathbb{S}_{\mathrm{UV}}^{(0,0)} + \mathbb{Q}^{(1,0)}\mathbb{S}_{\mathrm{UV}}^{(-1,1)} = 0, \tag{3.27}$$

which translates into

$$\widetilde{\mathsf{d}}S_{\mathrm{UV}} = \frac{1}{2}\int_{\mathbb{X}} d^4x\sqrt{g}\Psi^{\mu\nu}\mathcal{Q}\Lambda^{\mathrm{UV}}_{\mu\nu}, \tag{3.28}$$

aligning with our above comment that $\mathcal{Q}\Lambda^{\mathrm{UV}}_{\mu\nu} = T^{\mathrm{UV}}_{\mu\nu}$. Continuing, we find

$$\mathbb{Q}^{(-1,2)}\mathbb{S}_{\mathrm{UV}}^{(0,0)} + \mathbb{Q}^{(0,1)}\mathbb{S}_{\mathrm{UV}}^{(-1,1)} + \mathbb{Q}^{(1,0)}\mathbb{S}_{\mathrm{UV}}^{(-2,2)} = 0. \tag{3.29}$$

This is the most complicated of the relations we will find, and tells us that

$$(\mathsf{K} + \Delta_H)S_{\mathrm{UV}} = \widetilde{\mathsf{d}}\left(\frac{1}{2}\int_{\mathbb{X}} d^4x\sqrt{g}\Psi^{\mu\nu}\Lambda^{\mathrm{UV}}_{\mu\nu}\right) - \int_{\mathbb{X}} d^4x\sqrt{g}\left(\Phi^\sigma\mathcal{Q}Z^{\mathrm{UV}}_\sigma + \Psi^{\mu\sigma}\Psi^\nu{}_\sigma\mathcal{Q}\Upsilon^{\mathrm{UV}}_{\mu\nu}\right). \tag{3.30}$$

While not eminently helpful, it is good to have the ability to write transformations as $\mathcal{Q}$-exact objects, as their expectation values will vanish. Next, we turn to

$$\mathbb{Q}^{(-1,2)}\mathbb{S}_{\mathrm{UV}}^{(-1,1)} + \mathbb{Q}^{(0,1)}\mathbb{S}_{\mathrm{UV}}^{(-2,2)} = 0, \tag{3.31}$$

which gives

$$\frac{1}{2}\int_{\mathbb{X}} d^4x\sqrt{g}\Psi^{\mu\nu}\mathsf{K}\Lambda^{\mathrm{UV}}_{\mu\nu} = -\widetilde{\mathsf{d}}\left(\int_{\mathbb{X}} d^4x\sqrt{g}\left(\Phi^{\sigma}Z^{\mathrm{UV}}_{\sigma} + \Psi^{\mu\sigma}\Psi^{\nu}{}_{\sigma}\Upsilon^{\mathrm{UV}}_{\mu\nu}\right)\right). \tag{3.32}$$

Finally, in the highest gravity degree, we note

$$\mathbb{Q}^{(-1,2)}\mathbb{S}^{(-2,2)}_{\mathrm{UV}} = \int_{\mathbb{X}} d^4x\sqrt{g}\Phi^{\sigma}\mathsf{K}Z^{\mathrm{UV}}_{\sigma} = 0. \tag{3.33}$$

Another common computation will be $\mathbb{Q}^{(0,1)}\mathbb{S}^{(-1,1)}_{\mathrm{UV}}$, so it is worth conducting now. We have

$$\mathbb{S}^{(-1,1)}_{\mathrm{UV}} = -\frac{1}{2}\int_{\mathbb{X}} d^4x\sqrt{g}\Psi^{\mu\nu}\Lambda_{\mu\nu} = -\widetilde{\mathsf{d}}(V^{\mathrm{Loc}}_{\mathrm{UV}} + V^{\mathrm{Pro}}_{\mathrm{UV}} + V^{\mathrm{Pot}}_{\mathrm{UV}}), \tag{3.34}$$

So that

$$\widetilde{\mathsf{d}}\mathbb{S}^{(-1,1)} = -\widetilde{\mathsf{d}}^2(V^{\mathrm{Loc}}_{\mathrm{UV}} + V^{\mathrm{Pro}}_{\mathrm{UV}} + V^{\mathrm{Pot}}_{\mathrm{UV}}) \tag{3.35}$$

Term by term, we have

$$\begin{aligned}\widetilde{\mathsf{d}}^2\Psi^{\mathrm{Loc}} = \frac{1}{2g_0^2}\int_{\mathbb{X}} d^4x\Big[&\mathcal{L}_{\Phi}(\sqrt{g}g^{\mu\rho}g^{\nu\sigma})\mathrm{Tr}[\frac{1}{2}\chi_{\mu\nu}F_{\rho\sigma}] - \frac{1}{4}\Psi^{\rho\sigma}\Psi_{\rho[\mu}\sqrt{g}\mathrm{Tr}[\chi_{\nu]\sigma}F^{\mu\nu,+}] \\ &-\sqrt{g}\mathrm{Tr}[\frac{1}{2}\chi^{\mu\nu}((\mathcal{L}^{(A)}_{\Phi}F^{+})_{\mu\nu} - (\mathcal{L}^{(A)}_{\Phi}F)^{+}_{\mu\nu})]\Big], \end{aligned}\tag{3.36}$$

$$\widetilde{\mathsf{d}}^2\Psi^{\mathrm{Pro}} = -\frac{1}{2g_0^2}\int_{\mathbb{X}} d^4x\mathcal{L}_{\Phi}(\sqrt{g}g^{\mu\rho})\mathrm{Tr}\left[2\lambda D_{\mu}\psi_{\rho}\right], \tag{3.37}$$

$$\widetilde{\mathsf{d}}^2\Psi^{\mathrm{Pot}} = -\frac{1}{2g_0^2}\int_{\mathbb{X}} d^4x\mathcal{L}_{\Phi}(\sqrt{g})\mathrm{Tr}\left[2\eta[\phi,\lambda]\right], \tag{3.38}$$

which gives us the result. Now, let us return to $\mathsf{Z}[g,\Psi,\Phi]$

3.4 Expanding $\mathsf{Z}[g,\Psi,\Phi]$

Due to the exponential, $\mathsf{Z}[g,\Psi,\Phi]$ is a polyform in $H_{\mathsf{Diff}_+(\mathbb{X})}(\mathsf{Met}(\mathbb{X}))$. We would like to work with objects of fixed degree, and therefore, we expand the differential in gravity degree. This takes the form

$$\begin{aligned}\mathsf{Z}[g,\Psi,\Phi] &= \int [d\mathrm{VM}]\left(1 - \mathbb{S}_{\mathrm{UV}}^{(-1,1)} + \left((\mathbb{S}_{\mathrm{UV}}^{(-1,1)})^2 - \mathbb{S}_{\mathrm{UV}}^{(-2,2)}\right) + \cdots\right) e^{-S_{\mathrm{UV}}} \\ &= \langle \mathbb{1}\rangle_{\mathrm{UV}} - \langle \mathbb{S}_{\mathrm{UV}}^{(-1,1)}\rangle_{\mathrm{UV}} + \langle (\mathbb{S}_{\mathrm{UV}}^{(-1,1)})^2 - \mathbb{S}_{\mathrm{UV}}^{(-2,2)}\rangle_{\mathrm{UV}} + \cdots \end{aligned} \tag{3.39}$$

We now come to an important point. An astute observer may cry foul, claiming that clearly the elements of odd gravity degree vanish, as they are the expectation values of an odd number of twisted vector multiplet fermions. For example, since the gravitinos are background fields, we have

$$\langle \mathbb{S}_{\mathrm{UV}}^{(-1,1)}\rangle_{\mathrm{UV}} = \frac{1}{2}\int_{\mathbb{X}} d^4x\sqrt{g(x)}\,\Psi^{\mu\nu}(x)\langle \Lambda^{\mathrm{UV}}_{\mu\nu}(x)\rangle_{\mathrm{UV}}, \tag{3.40}$$

where Λ^{UV} exclusively contains terms with one fermion. Standard lore dictates that such terms must vanish as they are Grassmann valued, but in this case the lore is wrong! To see this, suppose we have $b_1 = 1$, so the space $H^1(\mathbb{X})$ is one dimensional. In our twisted theory, the ψ are elements of $\Pi\Omega^1(\mathbb{X})$, and therefore our path integral measure contains a single zeromode ψ_0 associated to the generating element of $H^1(\mathbb{X})$. Thus, with an understanding of fermionic integral, our expectation value will contain a factor of the form

$$\int [d\psi_0]\psi_0 = 1. \tag{3.41}$$

Of course, if $b_1 > 1$, there are an insufficient number of insertions to "soak" up all the zeromodes and thus the expectation would vanishes.[42] Inspecting the explicit terms in Λ^{UV}, we see that the terms with χ can lead to non-zero contributions when $b_2^+ = 1$, and since $H^0(\mathbb{X})$ is always one dimensional, the term with η always stands a chance of surviving. Hence, we are not upset by correlation functions with an odd number of fermionic insertions and happily continue on. We refer the reader to [63, 64] for a more thorough treatment of this point in the context of the original Donaldson-Witten invariants.

Moving along, we define

$$\mathsf{Z}[g, \Psi, \Phi] = \sum_{m=0}^{\infty} \mathsf{Z}^{[m]}, \tag{3.42}$$

where the sum is over increasing gravity degree. Since d is homogeneous of degree one, we conclude that

$$\mathsf{d}\mathsf{Z}^{[m]} = 0 \quad \text{for all } m. \tag{3.43}$$

We will soon go through the first few degrees to see how this works. Throughout, we will make use of the fact that the expectation value of any $\mathcal{Q}$-exact term is zero, that is, we have, for all $\mathbb{A} \in \mathbb{E}$,

$$\int [d\mathrm{VM}] \mathcal{Q}\mathbb{A} e^{-S^{\mathrm{UV}}} = \langle \mathcal{Q}\mathbb{A} \rangle_{\mathrm{UV}} = 0. \tag{3.44}$$

[42] The nonvanishing of fermionic expectation values is not restricted to the twisted theory. Indeed, the one point insertion of a free massive Majorana fermion on a torus with RR boundary conditions (periodic in both chiralities) is nonvanishing as can be gleaned by reading between the line of Section 12.4.1 of [29].

Consider our stated identity

$$\mathsf{d}\langle \mathbb{A} \rangle_{\mathrm{UV}} = \langle \mathbb{Q}\mathbb{A} \rangle_{\mathrm{UV}}. \tag{3.45}$$

Note that d is a differential of bidegree $(0,1)$ while $\mathbb{Q}$ has pieces of bidegrees $(1,0)$, $(0,1)$ and $(-1,2)$. Suppose that $\mathbb{A}$ has bidegree (p,q). The right hand side of the above then has an integrand with pieces of bidegrees $(p+1,q)$, $(p,q+1)$, and $(p-1,q+2)$. Naturally, since the expectation value of $\mathcal{Q}$-exact pieces vanishes, we are left with a piece of bidegree $(p,q+1)$ and one of bidegree $(p-1,q+2)$. Meanwhile the left hand side is only of bidegree $(p,q+1)$. Therefore, the $(p-1,q+2)$ term must vanish on its own. From a imprecise perspective, since $\mathbb{Q}^{(-1,2)}$ is an artifact of the vector supersymmetry (ignoring Δ_H), the desired statement may be an analogous to $\langle \mathcal{Q}\mathbb{A} \rangle_{\mathrm{UV}} = 0$, but for the vector supersymmetry. Nevertheless, this ignores the fact that we have fixed ourselves to a symmetric gravitino background and do not work with a rigid vector supersymmetry, which is not guarenteed to exist on an arbitrary smooth four manifold. We will see that these terms of bidegree $(p-1,q+2)$ vanish in $\mathsf{d}\mathsf{Z}^{[0]}$, but it is not as easily seen in $\mathsf{d}\mathsf{Z}^{[m>0]}$. Given the strength of our chain map argument, we conjecture that there is a series of Ward identities which will save the day.

3.4.1 $\mathsf{Z}^{[0]}$

At degree zero, we have the usual Donaldson-Witten partition function of

$$\mathsf{Z}^{[0]}[g] = \int [d\mathrm{VM}]\, e^{-S_{\mathrm{UV}}} = \langle \mathbb{1} \rangle_{\mathrm{UV}}. \tag{3.46}$$

The closure goes as

$$\begin{aligned} \mathsf{d}\mathsf{Z}^{[0]}[g] &= \int [d\mathrm{VM}](-\mathbb{Q}S_{\mathrm{UV}})e^{-S_{\mathrm{UV}}}, \\ &= -\int [d\mathrm{VM}](\mathcal{Q}S_{\mathrm{UV}} + \widetilde{\mathsf{d}}S^{\mathrm{UV}} + (\mathsf{K}+\Delta_H)S_{\mathrm{UV}})e^{-S_{\mathrm{UV}}} \end{aligned} \tag{3.47}$$

The first term clearly vanishes. Next, using (3.28) and (3.30), we find

$$\begin{aligned} \mathsf{d}\mathsf{Z}^{[0]}[g] = \int_{\mathbb{X}} d^4x\sqrt{g}\langle \mathcal{Q}\Lambda^{\mathrm{UV}}_{\mu\nu}\rangle \Psi^{\mu\nu} + \langle \widetilde{\mathsf{d}}\mathbb{S}^{(-1,1)}_{\mathrm{UV}}\rangle \\ + \int_{\mathbb{X}} d^4x\sqrt{g}\left(\Phi^{\sigma}\langle \mathcal{Q}Z^{\mathrm{UV}}_{\sigma}\rangle + \Psi^{\mu\sigma}\Psi^{\nu}{}_{\sigma}\langle \mathcal{Q}\Upsilon^{\mathrm{UV}}_{\mu\nu}\rangle\right) \end{aligned} \tag{3.48}$$

Again dropping the $\mathcal{Q}$-exact terms, and summing the term of (3.36)-(3.38), we find

$$\begin{aligned} \mathsf{d}\mathsf{Z}^{[0]}[g] = \frac{1}{2g_0^2}\int [d\mathrm{VM}]\int_{\mathbb{X}} d^4x \Big[&\mathcal{L}_{\Phi}(\sqrt{g}g^{\mu\rho}g^{\nu\sigma})\mathrm{Tr}[\frac{1}{2}\chi_{\mu\nu}F_{\rho\sigma}] - \sqrt{g}\mathrm{Tr}[\frac{1}{2}\chi^{\mu\nu}((\mathcal{L}^{(A)}_{\Phi}F^{+})_{\mu\nu} \\ &-(\mathcal{L}^{(A)}_{\Phi}F)^{+}_{\mu\nu})] - \frac{1}{4}\Psi^{\rho\sigma}\Psi_{\rho[\mu}\sqrt{g}\mathrm{Tr}[\chi_{\nu]\sigma}F^{\mu\nu,+}] \\ &-\mathcal{L}_{\Phi}(\sqrt{g}g^{\mu\rho})\mathrm{Tr}\left[2\lambda D_{\mu}\psi_{\rho}\right] - \mathcal{L}_{\Phi}(\sqrt{g})\mathrm{Tr}\left[2\eta[\phi,\lambda]\right]\Big] e^{-S^{\mathrm{UV}}} \end{aligned} \tag{3.49}$$

Utilizing Fubini's theorem with our vector multiplet and spacetime integral, we can harmlessly pass the Lie derivatives $\mathcal{L}_{\Phi}$ over all the integrated vector multiplet fields, which become fixed values. This results in a total derivative over $\mathbb{X}$ which vanishes as our manifold is without boundary. Hence, we find a remaining

$$\mathsf{d}\mathsf{Z}^{[0]}[g] = -\frac{1}{4}\int_{\mathbb{X}} d^4x\sqrt{g}\Psi^{\rho\sigma}\Psi_{\rho[\mu}\langle \mathrm{Tr}[\chi_{\nu]\sigma}F^{\mu\nu,+}]\rangle_{\mathrm{UV}}, \tag{3.50}$$

which we recognize as stemming from the curvature of the projected connection. Our final maneuver is to use a equation of motion of the original Donaldson-Witten theory, particularly the famous instanton equation $F_A^+ = 0$. Thus we conclude, as expected, that

$$\mathsf{d}\mathsf{Z}^{[0]}[g] = 0. \tag{3.51}$$

We note that we have to work a pinch harder to show this than in the original case of $\mathsf{Z}_{\mathrm{DW}}[g]$. This is due to the beforementioned $(p-1, q+2)$ term (here manifesting as $(-1, 2)$).

3.4.2 $\mathsf{Z}^{[1]}$

The only degree one gravity field is Ψ, so we have

$$\begin{aligned}\mathsf{Z}^{[1]}[g, \Psi] &= \int [d\mathrm{VM}] \left(\frac{1}{2} \int_{\mathbb{X}} d^4x \sqrt{g} \Psi^{\mu\nu} \Lambda^{\mathrm{UV}}_{\mu\nu} \right) e^{-S_{\mathrm{UV}}} \\ &= -\frac{1}{2} \int_{\mathbb{X}} d^4x \sqrt{g(x)} \langle \Lambda^{\mathrm{UV}}_{\mu\nu}(x) \rangle_{\mathrm{UV}} \Psi^{\mu\nu}(x).\end{aligned} \tag{3.52}$$

The d-closure goes as

$$\begin{aligned}\mathsf{d}\mathsf{Z}^{[1]}[g, \Psi] = \int [d\mathrm{VM}] \Bigg(\mathbb{Q} \left(\frac{1}{2} \int_{\mathbb{X}} d^4x \sqrt{g} \Psi^{\mu\nu} \Lambda^{\mathrm{UV}}_{\mu\nu} \right) \\ - \left(\frac{1}{2} \int_{\mathbb{X}} d^4x \sqrt{g} \Psi^{\mu\nu} \Lambda^{\mathrm{UV}}_{\mu\nu} \right) (\mathbb{Q} S_{\mathrm{UV}}) \Bigg) e^{-S_{\mathrm{UV}}}.\end{aligned} \tag{3.53}$$

Here, we can immediately drop the terms that are $\mathcal{Q}$-exact. Further, from the reasoning of the last section, the expectation value of $\widetilde{\mathsf{d}}\mathbb{S}^{(-1,1)}_{\mathrm{UV}}$ also vanishes. This leaves us with

$$\mathsf{d}\mathsf{Z}^{[1]}[g, \Psi] = \int [d\mathrm{VM}] \Big((\mathsf{K} + \Delta_H) \mathbb{S}^{(-1,1)}_{\mathrm{UV}}$$

$$- \left(\frac{1}{2} \int_{\mathbb{X}} d^4x \sqrt{g} \Psi^{\mu\nu} \Lambda^{\rm UV}_{\mu\nu} \right) (\widetilde{\mathsf{d}} + (\mathsf{K} + \Delta_H)) S_{\rm UV} \Bigg) e^{-S_{\rm UV}} \tag{3.54}$$

Using (3.28) we see that one term is the expectation value of $\mathcal{Q}(\mathbb{S}^{(-1,1)}_{\rm UV}\mathbb{S}^{(-1,1)}_{\rm UV})$, which vanishes. This leaves us with

$$\mathsf{dZ}^{[1]}[g,\Psi] = \int [d\mathrm{VM}] \left(\mathsf{K}\mathbb{S}^{(-1,1)}_{\rm UV} - \left(\frac{1}{2} \int_{\mathbb{X}} d^4x \sqrt{g} \Psi^{\mu\nu} \Lambda^{\rm UV}_{\mu\nu} \right) (\mathsf{K} + \Delta_H) S_{\rm UV} \right) e^{-S_{\rm UV}}. \tag{3.55}$$

At present, neither of these two terms obviously vanishes. Explicitly, they are

$$\begin{aligned} \langle \mathsf{K}\mathbb{S}^{(-1,1)} \rangle = \int [d\mathrm{VM}] \Bigg(-\frac{1}{4g_0^2} \int_{\mathbb{X}} d^4x \sqrt{g} \Psi^{\mu\nu} \mathrm{Tr}\, [&4(D_\mu\lambda)\Phi^\sigma F_{\sigma\nu} - 2g_{\mu\nu}(D_\sigma\lambda)\Phi^\rho F_\rho{}^\sigma \\ &+ 2g_{\mu\nu}\Phi^\sigma(D_\sigma\lambda)[\phi,\lambda] + 2g_{\mu\nu}\eta[\Phi^\sigma\psi_\sigma,\lambda]] \Bigg) e^{-S_{\rm UV}}, \end{aligned} \tag{3.56}$$

and

$$\begin{aligned} \langle \mathbb{S}^{(-1,1)}_{\rm UV} (\mathsf{K} + \Delta_H) S^{\rm UV} \rangle = \Bigg\langle & \left(\frac{1}{2} \int_{\mathbb{X}} d^4x \sqrt{g} \Psi^{\mu\nu} \Lambda^{\rm UV}_{\mu\nu} \right) \\ & \times \frac{1}{2g_0^2} \int_{\mathbb{X}} d^4y \sqrt{g} \mathrm{Tr} \Bigg[\frac{1}{4} \Psi^{\rho\sigma} \Psi_{\rho[\mu}\chi_{\nu]\sigma} (F^{\mu\nu}_+ + D^{\mu\nu}) \\ & + \frac{1}{4} \Psi^{\rho\sigma} \Psi_\rho{}^\mu \chi_{\mu\nu} D^\nu{}_\sigma + \frac{1}{4}\chi_{\mu\nu}[\Phi^\sigma\psi_\sigma, \chi^{\mu\nu}] \\ & - 2\Phi^\sigma(D_\sigma\lambda) D_\mu\psi^\mu + 2\eta D_\mu(\Phi^\sigma F_\sigma{}^\mu) \\ & + 4\lambda[\Phi^\sigma F_{\sigma\mu}, \psi^\mu] + 2\lambda D_\mu D^\mu (\Phi^\sigma\psi_\sigma) \\ & + 4[\Phi^\sigma\psi_\sigma,\lambda][\phi,\lambda] - 2\Phi^\sigma(D_\sigma\lambda)[\phi,\eta] \\ & + 2\eta[\phi, \Phi^\sigma D_\sigma\lambda] \Bigg] \Bigg\rangle_{\rm UV}. \end{aligned} \tag{3.57}$$

With the assurance that $\mathsf{d}\mathsf{Z}^{[1]} = 0$, we conjecture that these two terms must vanish due to a nontrivial Ward identity, explicitly

$$\langle \mathsf{K}\mathbb{S}_{\mathrm{UV}}^{(-1,1)} \rangle_{\mathrm{UV}} = \langle \mathbb{S}_{\mathrm{UV}}^{(-1,1)} (\mathsf{K} + \Delta_H) S_{\mathrm{UV}} \rangle_{\mathrm{UV}}. \tag{3.58}$$

3.4.3 $\mathsf{Z}^{[m>1]}$

In higher degree, we take it as a fact that

$$\mathsf{d}\mathsf{Z}^{[m>1]} = 0. \tag{3.59}$$

Explicitly computing the variation on the left hand side one can obtain an infinite number of conjectural non-trivial Ward identities. We leave the exploration of said identities to a future project.

4 The Future

We have provided the construction of the Cartan model of equivariant cohomology for $H_{\mathbb{G}}(\mathbb{M})$ and shown its connection to truncated twisted supergravity on a symmetric gravitino background. In addition, we provided both a UV and IR action principle, allowing for a construction of our invariants $\mathsf{Z}[g, \Psi, \Phi]$ which can be understood as elements of $H_{\mathbb{G}}(\mathbb{X})$. This is chapter zero in a unwritten longer story. Herein we present prologues to various potential sequels.

4.1 The Observables

We expect that there will be generalizations of the "n-observable" of Donaldson-Witten theory to family observables.[43] From this perspective the above "partition function" plays the role of the trivial observable. So call higher degree observables will be associated to a homology classes of the four manifold $\mathbb{X}$, giving a generalization of the Donaldson map μ_{D}. In the same way that we conceived the partition function $\mathsf{Z}_{\mathrm{DW}}[g]$ as the gravity degree zero part of an expansion of $\mathsf{Z}[g, \Psi, \Phi]$, we hope to realize the Donaldson-Witten polynomial invariants $\mathsf{Z}_{\mathrm{DW}}[g, p, \Sigma]$ in (0.152) as the gravity degree zero part of a similar expansion. Naturally things are not so simple. Instead we present three different perspective on where these family observables may be hiding.

4.1.1 Naive Family Invariants

On the gauge Cartan base fields, we have

$$\begin{aligned} \mathbb{Q}A_\mu &= \psi_\mu, & \mathbb{Q}\psi_\mu &= -D_\mu\phi + \Phi^\sigma F_{\sigma\mu}, \\ \mathbb{Q}\phi &= -\Phi^\sigma\psi_\sigma. \end{aligned} \tag{4.1}$$

This leads us to the rather interesting "ascent & descent" equation on the original n-observables of (0.127)-(0.131)

$$\mathbb{Q}\mathcal{O}^{(n)} = d\mathcal{O}^{(n-1)} + \iota_\Phi\mathcal{O}^{(n+1)}, \tag{4.2}$$

[43] We are in part led to this conclusion from the math literature on family invariants cf. [57, 49]

where ι_Φ is the interior derivative along the vector field Φ. Thus, for Σ_n without boundary, we have

$$\mathbb{Q}\mathcal{O}^{(n)}(\Sigma_n) = \int_{\Sigma_n} \iota_\Phi \mathcal{O}^{(n+1)}. \tag{4.3}$$

This indicates that $\mathcal{O}^{(n)}(\Sigma_n)$ are, in general, not objects in the cohomology of $H_{\mathbb{G}}(\mathbb{M})$. Further, recall our chain map argument in (3.9), which states that

$$\mathsf{d}\langle \mathbb{A} \rangle_{\mathrm{UV}} = \langle \mathbb{Q}\mathbb{A} \rangle_{\mathrm{UV}} \tag{4.4}$$

for any $\mathbb{A} \in \mathbb{E}_g$. Thus, since $\mathbb{Q}\mathcal{O}^{(n)}(\Sigma_n) \neq 0$, we find that it is not immediately obvious that $\mathsf{d}\langle \mathcal{O}^{(n)}(\Sigma_n) \rangle = 0$. Hence, due to the gravity degree two fields Φ we have no reason to suspect that $\langle \mathcal{O}^{(n)}(\Sigma_n) \rangle_{\mathrm{UV}}$ is an element of $H_{\mathsf{Diff}_+(\mathbb{X})}(\mathsf{Met}(\mathbb{X}))$. In the original theory, prior to working with the equivariant differential d, these expectation values were seen to be formally independent of the metric and thus elements of $H^0_{\mathsf{Diff}_+(\mathbb{X})}(\mathsf{Met}(\mathbb{X}))$.

Nevertheless, one is naturally led to define the *naive generalized n-observable* as

$$\mathsf{O}^{(n)}[g, \Psi, \Phi, \Sigma_n] = \int [d\mathrm{VM}] \mathcal{O}^{(n)}(\Sigma_n) e^{-\mathbb{S}_{\mathrm{UV}}}. \tag{4.5}$$

Focusing on the case of $n = 0$, we have

$$\mathsf{O}^{(0)}[g, \Psi, \Phi, p_0] = \int [d\mathrm{VM}] \frac{1}{2} \mathrm{Tr}[\phi^2(p_0)] e^{-\mathbb{S}_{\mathrm{UV}}}. \tag{4.6}$$

Expanding, at degree zero we have the original expectation value of the 0-observable, namely

$$\mathsf{O}^{(0)[0]}[g, p_0] = \langle \mathcal{O}^{(0)}(p_0) \rangle_{\mathrm{UV}} = \int [d\mathrm{VM}] \frac{1}{2} \mathrm{Tr}[\phi^2(p_0)] e^{-S_{\mathrm{UV}}}. \tag{4.7}$$

This is known to be formally independent of the metric for $b_2^+ > 1$, and thus it should be the case that $\mathsf{dO}^{(0)[0]}[g, p_0] = 0$, but due to the second term in our "ascent & descent" equations (4.2), we instead find

$$\mathsf{dO}^{(0)[0]}[g, p_0] = -\Phi^\sigma \langle \mathrm{Tr}[\psi_\sigma(p_0)\phi(p_0)]\rangle_{\mathrm{UV}}, \tag{4.8}$$

which in principle does not vanish for all $\mathbb{X}$. This, at present, remains a mystery and stems from the bidegree $(-1, 2)$ part of our differential $\mathbb{Q}$.

4.1.2 Donaldson's Construction

Turning to an alternative direction, recall that for a fixed instanton number k, the Donaldson invariants are counting, with signs, the numbers of instantons that can be put on the manifold $\mathbb{X}$. In the Mathai-Quillen formalism, we understand these invariants as the integral of equivariant cohomology classes $H_{\mathcal{G}}(\mathcal{A}(P))$ over $\mathcal{M}_{k,g}$. Therefore, if the virtual dimension of $\mathcal{M}_{k,g}$ is greater than zero, say $d > 0$, the invariants vanishes unless one inserts appropriate n-observables, so that their total degree as elements of $H_{\mathcal{G}}(\mathcal{A}(P))$ equals d. We can then understand the n-observables as objects which "lower" the dimension of the set of allowed instantons on $\mathbb{X}$.

On the other hand, suppose that the virtual dimension of the moduli space $\mathcal{M}_{k,g}$ is $d < 0$. This tells us that, with this fixed k, $\mathbb{X}$ will generically not support any instanton solutions. Nevertheless, considered over a d parameter family of metrics γ_d, there will generically be a finite number of metrics in the family on which $\mathbb{X}$ does support a finite number of instantons. In principle, the integral of $\mathsf{Z}^{[d]}[g, \Psi, \Phi]$ over γ_d will count, again with signs, the number of instantons that exist somewhere in the family. Thus, a family of metrics will "raise" the dimension of the moduli space.

One known sketch of the combination of these ideas is found in [24], which we resurrect here. Suppose that the dimension of $\mathcal{M}_{k,g}$ is generically $d+2n$ with $d<0$. Moreover, let us take γ_d to be parameterized by $t \in B$ for some closed, compact subspace $B \subset \mathbb{R}^d$. We can then define the family moduli space associated to γ_d as

$$\mathcal{M}_{k,\gamma_d} = \left\{ ([A],t) \in \mathcal{A}/\mathcal{G} \times B \,\middle|\, [A] \in \mathcal{M}_{k,g_t} \right\}. \tag{4.9}$$

Next, consider a generic surface Σ in $\mathbb{X}$. Any irreducible connection which satisfies $F_A^+ = 0$ will remain irreducible on Σ, and thus we have the restriction map

$$R_\Sigma : \mathcal{M}_{k,g} \longrightarrow (\mathcal{A}^*/\mathcal{G})_\Sigma, \tag{4.10}$$

where $(\mathcal{A}^*/\mathcal{G})_\Sigma$ is the space of irreducible connections on Σ. Now, our equivariant cohomology class $\mathcal{O}^{(2)}(\Sigma)$ is pulled back from $(\mathcal{A}^*/\mathcal{G})_\Sigma$ via R_Σ. In $(\mathcal{A}^*/\mathcal{G})_\Sigma$ we can choose a generic codimension two submanifold which represents $\mathcal{O}^{(2)}(\Sigma)$ before it was pulled back. We call the preimage under R_Σ of this codimension two submanifold $V_\Sigma \subset \mathcal{M}_{k,g}$. Donaldson then defines an invariant $\sigma[\gamma_d, [\Sigma_1], \ldots, [\Sigma_d]]$ which counts the number of intersections $\mathcal{M}_{k,\gamma_d} \cap V_{\Sigma_1} \cap \cdots \cap V_{\Sigma_d}$. It is further claimed that this invariant defines an cohomology class of $H^d(\mathsf{Met}(\mathbb{X})/\mathsf{Diff}_+(\mathbb{X}))$.[44]Here, we see an invariant where both observables and families conspire to arrive at a zero dimension set of permissible instantons on $\mathbb{X}$. At present there is no known QFT representation of this construction.

[44]Technically, it is an element of the twisted cohomology $H^d(\mathsf{Met}(\mathbb{X})/\mathsf{Diff}_+(\mathbb{X}), \Pi)$, where Π is the local coefficient system over $\mathsf{Met}(\mathbb{X})/\mathsf{Diff}_+(\mathbb{X})$ corresponding to the representation of the $\mathsf{Diff}_+(\mathbb{X})$ on the multilinear, $\mathbb{Z}_2$-valued functions in the homology of $\mathbb{X}$.

4.1.3 Universal Chern Class

Let us begin by recalling the BRST model of (1.80)-(1.87). Specializing to the gauge half of the model, we have

$$d_{\mathcal{C},\mathcal{G}}A_\mu = \psi_\mu + D_\mu c, \qquad d_{\mathcal{C},\mathcal{G}}\psi_\mu = -D_\mu\phi - [c,\psi_\mu], \tag{4.11}$$

$$d_{\mathcal{C},\mathcal{G}}c = \phi - \frac{1}{2}[c,c], \qquad d_{\mathcal{C},\mathcal{G}}\phi = [\phi, c]. \tag{4.12}$$

In [6], the n-observables were collectively identified as the second Chern class of a *universal bundle*. On this bundle there is a *universal connection* given by the polyform

$$\mathbb{A}_{\mathcal{G}} = A + c. \tag{4.13}$$

Further, identifying the differential of this bundle as the sum of the exterior derivative of $\mathbb{X}$ and the BRST differential, we have the *universal curvature* given by

$$\mathbb{F}_{\mathcal{G}} = (d + d_{\mathcal{C},\mathcal{G}})A_{\mathcal{G}} + \frac{1}{2}[\mathbb{A}_{\mathcal{G}}, \mathbb{A}_{\mathcal{G}}] = F_A + \psi + \phi. \tag{4.14}$$

This bundle has a bigrading of form degree and gauge degree so that $[\mathbb{A}_{\mathcal{G}}] = 1$ and $[\mathbb{F}_{\mathcal{G}}] = 2$. We then have the BRST algebra relations equivalent to

$$(d + d_{\mathcal{C},\mathcal{G}})A_{\mathcal{G}} = \mathbb{F}_{\mathcal{G}} - \frac{1}{2}[\mathbb{A}_{\mathcal{G}}, \mathbb{A}_{\mathcal{G}}], \tag{4.15}$$

$$(d + d_{\mathcal{C},\mathcal{G}})\mathbb{F}_{\mathcal{G}} = -[\mathbb{A}_{\mathcal{G}}, \mathbb{F}_{\mathcal{G}}]. \tag{4.16}$$

For $G = \mathrm{SU}(2)$, we have the second Chern class for the universal bundle given by

$$\mathcal{O}_{\mathcal{G}} = \frac{1}{2}\mathrm{Tr}\mathbb{F}_{\mathcal{G}}^2, \tag{4.17}$$

which satisfies

$$(d + d_{\mathcal{C},\mathcal{G}})\mathcal{O}_{\mathcal{G}} = 0. \tag{4.18}$$

Spliting $\mathcal{O}_{\mathcal{G}}$ into form degrees, we have

$$\mathcal{O}_{\mathcal{G}} = \mathcal{O}^{(0)} - \mathcal{O}^{(1)} + \mathcal{O}^{(2)} - \mathcal{O}^{(3)} + \mathcal{O}^{(4)}, \tag{4.19}$$

where the choice of signs allows us to identify the above splitting as our n-observables densities. We then see (4.17) as equivalent to the descent equation. It is important to note that, even though $\mathbb{A}_{\mathcal{G}}$ contains the vertical field c, the universal curvative $\mathbb{F}_{\mathcal{G}}$ is entirely horizontal, and thus each of the n-observables, when integrated over the appropriate cycle, will give a basic class of $H_{\mathcal{G}}(\mathcal{A}(P))$.

For the diffeomorphism side of our Cartan model in [72, 73], and expanded upon in [81, 90] there is also a construction the universal Chern class for a theory of topological gravity. In these works, the gravititational fields are dynamical, which is distinctly *not* the settling of our theory. Nevertheless, there may be something to learn in the analysis. Here, the BRST transformations are now

$$d_{\mathcal{C},\mathsf{Diff}_+} g_{\mu\nu} = \Psi_{\mu\nu} - \nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu, \tag{4.20}$$

$$d_{\mathcal{C},\mathsf{Diff}_+} \Psi_{\mu\nu} = -\xi^\sigma(\nabla_\sigma \Psi_{\mu\nu}) - (\nabla_\mu \xi^\sigma) - (\nabla_\nu \xi^\sigma)\Psi_{\mu\sigma} + \nabla_\mu \Phi_\nu + \nabla_\nu \Phi_\mu, \tag{4.21}$$

$$d_{\mathcal{C},\mathsf{Diff}_+} \xi^\mu = \Phi^\mu - \xi^\sigma(\nabla_\sigma \xi^\mu), \tag{4.22}$$

$$d_{\mathcal{C},\mathsf{Diff}_+} \Phi^\mu = -\xi^\sigma(\nabla_\sigma \Phi^\mu) + \Phi^\sigma(\nabla_\sigma \xi^\mu). \tag{4.23}$$

Unlike the gauge case, our coordinates g on $\mathsf{Met}(\mathbb{X})$ are not connections, so our universal connection is more complicated. We have it as

$$\widetilde{\Gamma}^\mu{}_\nu = \Gamma^\mu{}_{\lambda\nu} dx^\lambda + \frac{1}{2}\Psi^\mu{}_\nu + \nabla_\nu \xi^\mu, \tag{4.24}$$

where $\Gamma^{\mu}{}_{\lambda\nu}$ is the standard Levi-Civita connection. This leads to the universal curvature of

$$\widetilde{R}^{\mu}{}_{\nu} = (d + d_{\mathcal{C},\mathsf{Diff}_+})\widetilde{\Gamma}^{\mu}{}_{\nu} + \widetilde{\Gamma}^{\mu}{}_{\lambda} \wedge \widetilde{\Gamma}^{\lambda}{}_{\nu} \tag{4.25}$$

$$= \frac{1}{2} R^{\mu}{}_{\mu\rho\sigma} dx^{\rho} \wedge dx^{\sigma} + (P^{\mu}{}_{\nu\rho} - R^{\mu}{}_{\nu\rho\lambda}\xi^{\lambda})dx^{\rho} \tag{4.26}$$

$$+ \frac{1}{2}(Q^{\mu}{}_{\nu} + 2P^{\mu}{}_{\nu\rho}\xi^{\rho} + R^{\mu}{}_{\nu\rho\lambda}\xi^{\rho}\xi^{\lambda}), \tag{4.27}$$

where

$$P_{\mu\nu\lambda} = \frac{1}{2}(\nabla_{\nu}\Psi_{\lambda\mu} - \nabla_{\mu}\Psi_{\lambda\nu}), \tag{4.28}$$

$$Q_{\mu\nu} = -\frac{1}{2}\Psi_{\nu\lambda}\Psi^{\lambda}{}_{\nu} - (\nabla_{\nu}\Phi_{\mu} - \nabla_{\mu}\Phi_{\nu}). \tag{4.29}$$

Here, we see that the universal curvature has dependence on the degree one generators of the Weil algebra. This is to be distinguished from the gauge case, where the curvature was happily horizontal. Nevertheless, we can form the second Chern class of the universal bundle and split it into form degree as

$$\mathcal{O}_{\mathsf{Diff}_+} = \widetilde{R}^{\mu}{}_{\nu} \wedge \widetilde{R}^{\nu}{}_{\mu} \tag{4.30}$$

$$= \mathcal{O}^{(0)}_{\mathsf{Diff}_+} + \mathcal{O}^{(1)}_{\mathsf{Diff}_+} + \mathcal{O}^{(2)}_{\mathsf{Diff}_+} + \mathcal{O}^{(3)}_{\mathsf{Diff}_+} + \mathcal{O}^{(4)}_{\mathsf{Diff}_+}, \tag{4.31}$$

and they indeed, by construction, satisfy a set of descent equations

$$d_{\mathcal{C},\mathsf{Diff}_+}\mathcal{O}^{(n)}_{\mathrm{Diff}} = -d\mathcal{O}^{(n-1)}_{\mathrm{Diff}} \tag{4.32}$$

and thus, if integrated over appropriate cycles, are closed under the differential $d_{\mathcal{C},\mathsf{Diff}_+}$. Nevertheless, outside of $\mathcal{O}^{(4)}_{\mathsf{Diff}_+}$, which is a well known topological invariant, each $\mathcal{O}^{(n)}_{\mathsf{Diff}_+}$

contains explicit dependence on ξ^μ and is therefore not an element of equivariant cohomology.

Inspired by these constructions, one could attempt to construct a universal bundle for our full $H_{\mathbb{G}}(\mathbb{M})$ model. In such a construction there would be observables that mix elements of the gauge Cartan model and the diffeomorphism Cartan model. At present this direction has not been fully explored.

***************Discussion of other options************

4.2 Computation

Another direction worthy of immediate attention is the computation of our invariants. This will be done in two steps. First, one must flow to the IR theory and repeat the analysis of the u-plane of [71] for each $\mathsf{Z}_{\mathrm{IR}}^{[n]}$, where we define

$$\mathsf{Z}_{\mathrm{IR}}[g,\Psi,\Phi] = \int [d\mathrm{VM}] e^{-\mathbb{S}_{\mathrm{IR}}}, \tag{4.33}$$

and likewise, split into gravity degrees as

$$\mathsf{Z}_{\mathrm{IR}}[g,\Psi,\Phi] = \sum_{i=0}^{\infty} \mathsf{Z}_{\mathrm{IR}}^{[n]}. \tag{4.34}$$

The $\mathbb{Q}$ closure of our theory assures us that these will equal the UV invariants of $\mathsf{Z}^{[n]}$. In computing the IR correlation functions, at the monopole and dyon point, we will need to also introduce the generalized action for a twisted hypermultiplet coupled to the dual $\mathrm{U}(1)_{\mathrm{D}}$ vector multiplet theory that is $\mathbb{Q}$-closed. Since the original action at this these points are a $\mathcal{Q}$-exact part plus a topological part, the generalization should be obvious, but does require an understanding of the hypermultiplets as modules over $H_{\mathbb{G}}(\mathbb{M})$. We also expect that the work of [50, 61] will be of use in any explicit calculations.

Assuming that the analysis can be carried out, the next step is to integrate $\mathsf{Z}_{\mathrm{IR}}^{[n]}$ over n-parameter families of metrics. Suppose we have some compact, connected subset $B \subset \mathbb{R}^n$ which parameterizes a continuous n-parameter family of metrics γ_B. We can then integrate over this family as

$$\int_B d^n t\, \mathsf{Z}_{\mathrm{IR}}^{[n]}[g(t), \Psi(t), \Phi(t)]. \tag{4.35}$$

In this context, the $\Psi_{\mu\nu}$ will realize their roles as forms on $\mathsf{Met}\mathbb{X}$. We have not yet established the roles of the degree two Φ fields in the integral. For the case of $n = 1$, this integral can be made more explicit. Suppose we have a non-trivial diffeomorphism $f \in \mathsf{Diff}_+(\mathbb{X})$ with $g_0 = f^* g_1$, for $g_0, g_1 \in \mathsf{Met}(\mathbb{X})$. We then take our family γ to be a path in $\mathsf{Met}(\mathbb{X})$ between g_0 and g_1. Our family invariant associated to this path is then

$$\begin{aligned} \int_\gamma dt \mathsf{Z}^{[1]}[g_t, \Psi_t] &= -\frac{1}{2}\int_0^1 dt \int_{\mathbb{X}} d^4x \sqrt{g_t(x)} \langle \Lambda_{\mathrm{IR}}^{\mu\nu}(x)\rangle_{\mathrm{IR}} \Psi_t^{\mu\nu}(x) \\ &= -\frac{1}{2}\int_0^1 dt \int d^4x \sqrt{g_t(x)} \langle \Lambda_{\mathrm{IR}}^{\mu\nu}(x)\rangle_{\mathrm{IR}} \frac{dg_{\mu\nu}^t(x)}{dt}. \end{aligned} \tag{4.36}$$

Given our construction of the integrand as a equivariant class in $H_{\mathsf{Diff}_+(\mathbb{X})}(\mathsf{Met}(\mathbb{X}))$, this should be independent of all metric data, namely the choice of γ, and the choice of g_0 and g_1, so long as $g_0 = f^* g_1$. Further, we expect that dependence on the choice of diffeomorphism f is only up to choice of which component of $\mathsf{Diff}_+(\mathbb{X})$ it is from, hence (4.36) should be an invariant associated to elements of $\pi_0(\mathsf{Diff}_+(\mathbb{X}))$.

Together, these two steps of flowing to the IR and integrating over a family of metrics will certainly meet with complications, but the results could potentially yield exciting new classes of invariants which will help reveal the relatively unexplored topology of $\mathsf{Diff}_+(\mathbb{X})$. It would also be of further interest to explore the connection of these invariants to automorphic forms, as it is known that there is a deep connection between the original Donaldson-Witten invariants and mock modular forms [13, 51, 60].

4.3 Wall Crossing

The original Donaldson-Witten invariants experience a phenomenon known as wall-crossing. This occurs for $b_2^+ = 1$, where $\mathsf{Z}_{\mathrm{DW}}[g]$ is only piecewise constant on $\mathsf{Met}(\mathbb{X})$. Generically, along a one-parameter family, there will be a point g_* where there is an anti-self-dual reducible connection, which will lead to a singularity in $\mathcal{M}_{k,g_*}$. Such an occurrence will lead to a change in $\mathsf{Z}_{\mathrm{DW}}[g]$ on either side of g_* in the family.

It is likewise the case that, for $b_2^+ = n$, a generic n-parameter family of metrics will have a finite number of points on which there is an anti-self-dual reducible connections. Thus, if one fixes a $n-1$-parameter family and varies the whole family along a transverse path, one will experience wall-crossing in the invariant $\mathsf{Z}^{[n-1]}$ integrated over the $n-1$-parameter family.

In the original analysis, wall-crossing is less a bug and more a feature being very helpful in both understanding and computing the Donaldson-Witten invariants. We hope that, if this higher order wall-crossing exists, it is likewise a boon to the theory.

5 Conclusion

We have journeyed through the wild world of four manifolds, delved the depths of supersymmetry, and gathered up arms of equivariant cohomology and supergravity to wage battle against the unknown. Nevertheless, this is just the beginning and there is much work to be done and far more questions than when we started. Still, one question shines above all other: what is the role of smooth structures in physics?

Appendices

A Conventions

A.1 Form Conventions

For a given p form $A_{(p)} \in \Omega^p(\mathbb{X})$, we write it locally as

$$A_{(p)} = \frac{1}{p!} A_{\mu_1 \cdot \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}. \tag{A.1}$$

Taking $B_{(q)} \in \Omega^q(\mathbb{X})$, our wedge product of $A_{(p)}$ and $B_{(q)}$ is given by

$$A_{(p)} \wedge B_{(q)} = \frac{1}{p!q!} A_{[\mu_1 \cdots \mu_p} B_{\mu_{p+1} \cdots \mu_{p+q}]} dx^{\mu_1} \wedge \cdots dx^{\mu_p} \wedge dx^{\mu_{p+1}} \wedge \cdots \wedge dx^{\mu_{p+q}}. \tag{A.2}$$

This defines

$$(A_{(p)} \wedge B_{(q)})_{\mu_1 \ldots \mu_p \mu_{p+1} \cdots \mu_{p+q}} = \frac{(p+q)!}{p!q!} A_{[\mu_1 \cdots \mu_p} B_{\mu_{p+1} \cdots \mu_{p+q}}. \tag{A.3}$$

The exterior derivative on $A_{(p)}$ gives a $(p+1)$ form $dA_{(p)} \in \Omega^{p+1}(\mathbb{X})$, given by

$$dA_{(p)} = \frac{1}{p!} \partial_{[\mu_1} A_{\mu_2 \cdots \mu_{p+1}]} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \wedge dx^{\mu_{p+1}} \tag{A.4}$$

$$= \frac{1}{(p+1)!} (dA_{(p)})_{\mu_1 \cdots \mu_{p+1}} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \wedge dx^{\mu_{p+1}}. \tag{A.5}$$

The Hodge star on $A_{(p)}$ gives a $(d-p)$ form, where d is the dimension of our manifold, almost always taken as $d = 4$. We have

$$\star A_{(p)} = \frac{1}{p!(d-p)!} \sqrt{g} \epsilon_{\mu_1 \cdots \mu_p \mu_{p+1} \cdots \mu_d} A^{\mu_1 \cdots \mu_p} dx^{\mu_{p+1}} \wedge \cdots \wedge dx^{\mu_d}. \tag{A.6}$$

Hence we obtain

$$A_{(p)} \wedge \star A_{(p)} = \frac{1}{p!} A_{\mu_1 \cdots \mu_p} A^{\mu_1 \cdots \mu_p} \sqrt{g} d^d x. \tag{A.7}$$

With these conventions, the various contractions of the field strength $F_A \in \Omega^2(\mathbb{X}, \mathsf{ad}\, P)$ we encounter satisfy

$$F_A \wedge \star F_A = \frac{1}{2} F_{\mu\nu} F^{\mu\nu} \sqrt{g} d^4 x, \tag{A.8}$$

$$F_A \wedge F_A = \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} d^4 x, \tag{A.9}$$

$$F_A^+ \wedge F_A^+ = \frac{1}{2} F_+^{\mu\nu} F_{\mu\nu}^+ \sqrt{g} d^4 x, \tag{A.10}$$

$$F_A^- \wedge F_A^- = -\frac{1}{2} F_-^{\mu\nu} F_{\mu\nu}^- \sqrt{g} d^4 x. \tag{A.11}$$

Further, for $\chi \in \Pi\Omega_g^{2,+}(\mathbb{X}, \mathsf{ad}\, P)$ and $D \in \Omega_g^{2,+}(\mathbb{X}, \mathsf{ad}\, P)$, we have

$$\chi \wedge D = \frac{1}{2} \chi_{\mu\nu} D^{\mu\nu} \sqrt{g} d^4 x. \tag{A.12}$$

A.2 Spinor Conventions

Our conventions for raising and lowering $\mathfrak{su}(2)_+$ and $\mathfrak{su}(2)_-$ indices follow the North-West South-East conventions. Hence,

$$\lambda^A = \epsilon^{AB} \lambda_B, \qquad \lambda_A = \lambda^A \epsilon_{AB}, \tag{A.13}$$

$$\overline{\lambda}^{\dot{A}} = \epsilon^{\dot{A}\dot{B}} \overline{\lambda}_{\dot{B}}, \qquad \overline{\lambda}_{\dot{A}} = \overline{\lambda}^{\dot{A}} \epsilon_{\dot{A}\dot{B}}, \tag{A.14}$$

where $\epsilon_{AB} = \epsilon^{AB} = -\epsilon_{\dot{A}\dot{B}} = -\epsilon^{\dot{A}\dot{B}}$, and $\epsilon_{12} = +1$. In particular, note that this means that, for any spinorial objects λ and ψ, we have

$$\psi^A \lambda_A = \psi^A \epsilon_{AB} \lambda^B = -\psi^A \epsilon_{BA} \lambda^B = -\epsilon_{BA} \psi^A \lambda^B = -\psi_B \lambda^B = -\psi_A \lambda^A, \tag{A.15}$$

and thus

$$\lambda^A \lambda_A = 0. \tag{A.16}$$

In addition, there are a number of useful identities, such as

$$\epsilon^{AB}\epsilon_{CB} = \delta^A_C, \qquad \epsilon^{AB}\epsilon_{AB} = 2, \qquad \text{and} \qquad \epsilon^{AB}\epsilon_{CD} = \delta^A_C\delta^B_D - \delta^A_D\delta^B_C. \tag{A.17}$$

Turning to the construction of our intertwiners, the standard Pauli matrices are given by

$$\tau^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \tau^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \tau^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{A.18}$$

We then write $\overrightarrow{\boldsymbol{\tau}} = (\tau_1, \tau_2, \tau_3)$, to denote the Pauli vector. We can then define the Euclidean σ-matrices as

$$(\sigma_a)^{A\dot{B}} = (\overrightarrow{\boldsymbol{\tau}}, -i\mathbb{1}_2)^{A\dot{B}} \qquad \text{and} \qquad (\widetilde{\sigma}_a)_{\dot{A}B} = (\overrightarrow{\boldsymbol{\tau}}, i\mathbb{1}_2)_{\dot{A}B}. \tag{A.19}$$

These matrices serve as intertwiners between the bi-spinorial representations and vector representation of SO(4). Since the majority of our analysis involves curved space, we choose to define these matrices with the frame index a, but when we are restricting our analysis to flat Euclidean space, we often abusively write the spacetime indices μ, ν, &c. When we are indeed working in curved space, we will often suppress the vielbein 1-form, and define

$$(\sigma_\mu)^{A\dot{B}} = e_\mu{}^a(\sigma_a)^{A\dot{B}}, \qquad \text{and} \qquad (\widetilde{\sigma}_\mu)_{\dot{A}B} = e_\mu{}^a(\widetilde{\sigma}_a)_{\dot{A}B} \tag{A.20}$$

These σ matrices satisfy the relations

$$\sigma_a\widetilde{\sigma}_b + \sigma_b\widetilde{\sigma}_a = 2\delta_{ab}\mathbb{1}_2, \qquad \widetilde{\sigma}_a\sigma_b + \widetilde{\sigma}_b\sigma_a = 2\delta_{ab}\mathbb{1}_2, \tag{A.21}$$

$$(\sigma_a)^\dagger = \widetilde{\sigma}_a, \qquad (\widetilde{\sigma}^a)^{\dot{A}A} = (\sigma^a)^{A\dot{A}}, \tag{A.22}$$

$$(\sigma_a)^{A\dot{B}}(\widetilde{\sigma}_a)_{\dot{C}D} = 2\delta^A_D\delta^{\dot{B}}_{\dot{C}} \tag{A.23}$$

At a notational level, we can use these σ-matrices to change between the frame indices a and the two component indices. For some object V_a we have

$$V_a = \frac{1}{\sqrt{2}}V_{A\dot{A}}(\sigma_a)^{A\dot{A}}, \qquad \text{and likewise} \qquad V_{A\dot{A}} = \frac{1}{\sqrt{2}}(\sigma^a)_{A\dot{A}}V_a. \tag{A.24}$$

This normalization is used exclusively in the Excursus of twisted supergravity, as it allows for the simple relation $e^\mu_{A\dot{A}}e^{B\dot{B}}_\mu = \delta^B_A\delta^{\dot{B}}_{\dot{A}}$. In Section 0, we change indices without the prefactor.

We also have the self-dual and anti-self-dual projectors, defined by

$$(\sigma^{ab})^A{}_B = \frac{1}{2}\left[(\sigma^a)^{A\dot{B}}(\widetilde{\sigma}^b)_{\dot{B}B} - (\sigma^b)^{A\dot{B}}(\widetilde{\sigma}^a)_{\dot{B}B}\right], \tag{A.25}$$

$$(\widetilde{\sigma}^{ab})_{\dot{A}}{}^{\dot{B}} = \frac{1}{2}\left[(\widetilde{\sigma}^a)_{\dot{A}C}(\sigma^b)^{C\dot{B}} - (\widetilde{\sigma}^b)_{\dot{A}C}(\sigma^a)^{C\dot{B}}\right], \tag{A.26}$$

which have the explicit form

$$\sigma^{ab} = \begin{bmatrix} 0 & i\tau^3 & -i\tau^2 & i\tau^1 \\ -i\tau^3 & 0 & i\tau^1 & i\tau^2 \\ i\tau^2 & -i\tau^1 & 0 & i\tau^3 \\ -i\tau^1 & -i\tau^2 & -i\tau^3 & 0 \end{bmatrix}, \qquad \text{and} \qquad \widetilde{\sigma}^{ab} = \begin{bmatrix} 0 & i\tau^3 & -i\tau^2 & -i\tau^1 \\ -i\tau^3 & 0 & i\tau^1 & -i\tau^2 \\ i\tau^2 & -i\tau^1 & 0 & -i\tau^3 \\ i\tau^1 & i\tau^2 & i\tau^3 & 0 \end{bmatrix}. \tag{A.27}$$

B Toy Weil Algebra

Here we present a simplified version of the Lie superalgebra of the Weil algebra, considering a semi-direct product of two finite dimensional Lie groups. Let G be a finite dimensional Lie group, with normal subgroup G_1 and subgroup G_2.[45] We then have the homomorphism $\phi : G_2 \to \mathrm{Aut}(G_1)$ taking $h \in G_2$ to ad_h i.e., $\phi(h)(g) = hgh^{-1}$ for $g \in G_1$. We can then construct the semi-direct product $G_1 \rtimes G_2$, whose group structure is given by

$$(g_1, h_1)(g_2, h_2) = (g_1 h_1 g_2 h_1^{-1}, h_1 h_2) \ . \tag{B.1}$$

In this group, the identity is simply the tuple of identities from the original Lie groups i.e., (e_1, e_2). Inverse are as $(g,h)^{-1} = (h^{-1}g^{-1}h, h^{-1})$. In addition, the splitting lemma indicates that we have the short exact sequence

$$1 \longrightarrow G_1 \xrightarrow{\ \iota\ } G_1 \rtimes G_2 \xrightarrow{\ \pi\ } G_2 \longrightarrow 1 \ , \tag{B.2}$$

and a group homomorphism $\psi : G_2 \to G_1 \rtimes G_2$ such that $\pi \circ \psi = \mathrm{id}_{G_2}$. Explicitly, we have $\iota(g) = (g, e_2)$, $\pi(g,h) = h$ and $\psi(h) = (e_1, h)$, so that we may write $\iota(\phi(h)(g)) = \psi(h)\iota(g)\psi(h^{-1})$.

Let $\mathsf{Lie}(G_1) = \mathfrak{g}_1$ and $\mathsf{Lie}(G_2) = \mathfrak{g}_1$ be the Lie algebra of G_1 and G_2 respectively. The Lie algebra of our semi-direct product $G_1 \rtimes G_2$ is govern by the derivation $d\phi : \mathfrak{g}_2 \to \mathrm{Der}(\mathfrak{g}_1)$ with explicitly form $d\phi(\beta)(\alpha) = \mathsf{ad}_\beta(\alpha) = [\beta, \alpha]$ for $\alpha \in \mathfrak{g}_1$ and $\beta \in \mathfrak{g}_2$, where the Lie bracket here is the one inherited from $\mathsf{Lie}(G) = \mathfrak{g}$, which contains both $\mathfrak{g}_1$ and $\mathfrak{g}_2$ as subalgebra. The fact that this is a derivation follows from the Jacobi

[45] This mimics our case of $G_1 = \mathcal{G}$ and $G_2 = \mathsf{Diff}_+(\mathbb{X})$, since for $f \in \mathsf{Diff}_+(\mathbb{X})$ and $g \in \mathcal{G}$, we indeed have $fgf^{-1} = f^*(g) \in \mathcal{G}$ i.e., $\mathcal{G}$ is a normal subgroup of the full group of $\mathcal{G} \rtimes \mathsf{Diff}_+(\mathbb{X})$.

identity. The Lie bracket on $\mathsf{Lie}(G_1 \rtimes G_2)$, distinguished from that of $\mathfrak{g}$ by a double bracket, is then defined as

$$\begin{aligned} [[(\alpha_1, \beta_1), (\alpha_2, \beta_2)]] &= ([\alpha_1, \alpha_2] + d\phi(\beta_1)(\alpha_2) - d\phi(\beta_2)(\alpha_1), [\beta_1, \beta_2]) \\ &= ([\alpha_1, \alpha_2] + [\beta_1, \alpha_2] - [\beta_2, \alpha_1], [\beta_1, \beta_2]) \ . \end{aligned} \tag{B.3}$$

Given that G_1 is normal in G, we have $[\beta, \alpha] \in \mathfrak{g}_1$ for all $\alpha \in \mathfrak{g}_1$ and $\beta \in \mathfrak{g}_2$. Note that the above Lie bracket is indeed antisymmetric and satisfies the Jacobi identity as a consequence of the Jacobi identity on each factor.

Let us consider a split basis of $\mathsf{Lie}(G_1 \rtimes G_2)$. Suppose that $\dim G_1 = N_1$ and $\dim G_2 = N_2$ and that $\{t_a^{(1)}\}$ and $\{t_i^{(2)}\}$ are bases of each $\mathfrak{g}_1$ and $\mathfrak{g}_2$ respectively, where $a = 1, \ldots, N_1$ and $i = N_1 + 1 \ldots, N_1 + N_2$. We will always use indices a, b, c for $\mathfrak{g}_1$ and i, j, k for $\mathfrak{g}_2$. Since both G_1 and G_2 are subgroups of the abstract Lie group G, we can use the structure constants of $\mathsf{Lie}(G)$ to define those of our subalgebra. This gives

$$[t_a^{(1)}, t_b^{(1)}] = f_{ab}{}^c t_c^{(1)} \ , \tag{B.4}$$

$$[t_i^{(2)}, t_j^{(2)}] = f_{ij}{}^k t_k^{(2)} \ , \tag{B.5}$$

where $f_{ab}{}^c$ and $f_{ij}{}^k$ are defined in G restricted to the relevant indices for G_1 and G_2. Next, we can specify a basis $\{T_A\}$ of $\mathfrak{g}_1 \rtimes \mathfrak{g}_2$, as $1 \leq A \leq N_1$ we have $T_A = (t_A^{(1)}, 0)$ for $1 \leq A \leq N_1$ and $T_A = (0, t_A^{(2)})$ for $N_1 + 1 \leq A \leq N_1 + N_2$. Then we can write

$$[[T_A, T_B]] = f_{AB}{}^C T_C \ , \tag{B.6}$$

where $f_{AB}{}^C = -f_{BA}{}^C$ and $f_{ij}{}^a = f_{ai}{}^j = 0$ for all $1 \leq a \leq N_1$ and $N_1 + 1 \leq i, j \leq N_1 + N_2$. This last constraint follows directly from the fact that G_1 is normal in G.

Note that when ${f_{ai}}^b \neq 0$, this structure is distinguished from a direct product, as such structure constants lead to the semi-direct product terms in the definition the Lie bracket on $\mathsf{Lie}(G_1 \rtimes G_2)$ above.

Our expression for the structure constants of $\mathsf{Lie}(G_1 \rtimes G_2)$ is a little opaque, and thus we give it a partial exegesis. The cases where $1 \leq a, b \leq N_1$, namely the indices for the $\mathfrak{g}_1$ subalgebra, and we have $[[T_a, T_b]] = {f_{ab}}^c T_c$. Likewise, for $N_1 + 1 \leq i, j \leq N_1 + N_2$, we have $[[T_i, T_j]] = {f_{ij}}^k T_k$. Were this a direct product of Lie groups, this would be the whole story, but we have a semi-direct product, so there are non-zero structure constants with mixed indices. Indeed, we have

$$[[T_a, T_i]] = [[(t_a^{(1)}, 0), (0, t_i^{(2)})]] = (-[t_i^{(2)}, t_a^{(1)}], 0) = (-{f_{ia}}^c t_c^{(1)}, 0) = {f_{ai}}^c T_c \neq 0 \ , \quad \text{(B.7)}$$

where we have used the fact that $[\mathfrak{g}_1, \mathfrak{g}_2] \in \mathfrak{g}_1$. Likewise, we have $[[T_i, T_a]] = {f_{ia}}^c T_c \neq 0$.

Next, let's turn to the Weil algebra $\mathcal{W}(\mathsf{Lie}(G_1 \rtimes G_2))$. It is defined as the Koszul algebra of the dual of $\mathsf{Lie}(G_1 \rtimes G_2)$, that is,

$$\mathcal{W}(\mathsf{Lie}(G_1 \rtimes G_2)) = S^*((\mathsf{Lie}(G_1 \rtimes G_2))^\vee) \otimes \Lambda^*((\mathsf{Lie}(G_1 \rtimes G_2))^\vee) \ . \qquad \text{(B.8)}$$

Given our basis $\{T_A\}$, we have an induced dual basis $\{\tilde{T}^A\}$ through the Killing forms on each algebra $\mathfrak{g}_1$ and $\mathfrak{g}_2$. In order for this to be a full basis for $(\mathfrak{g}_1 \rtimes \mathfrak{g}_2)^\vee$, we require the existence of a nondegenerate Killing form, which requires that $\mathfrak{g}_1 \rtimes \mathfrak{g}_2$ be semisimple. We can then take generators of this algebra as $\{\varphi^A\}$, the degree two elements of the symmetric algebra, and $\{\theta^A\}$, the degree one elements of the exterior algebra. With these generators, we have the Koszul operator $d_{\mathcal{W}}$ as a degree one differential on the algebra $\mathcal{W}$, defined as

$$d_{\mathcal{W}}\theta^A = \phi^A - \frac{1}{2}f_{BC}{}^A\theta^B\theta^C d_{\mathcal{W}}\phi^A = -f_{BC}{}^A\theta^B\phi^C \ , \tag{B.9}$$

where as always, repeated indices are summed. Taking the split basis, noting when the structure constants vanish, we have

$$d_{\mathcal{W}}\theta^a = \phi^a - \frac{1}{2}f_{bc}{}^a\theta^b\theta^c - f_{ic}{}^a\theta^i\theta^c \ , \qquad d_{\mathcal{W}}\theta^i = \phi^i - \frac{1}{2}f_{jk}{}^i\theta^j\theta^k \ , \tag{B.10}$$

$$d_{\mathcal{W}}\phi^a = -f_{bc}{}^a\theta^b\phi^c - f_{bk}{}^a\theta^b\phi^k - f_{jc}{}^a\theta^j\phi^c \ , \quad d_{\mathcal{W}}\phi^i = -f_{jk}{}^i\theta^j\phi^k \ . \tag{B.11}$$

Note that $d_{\mathcal{W}}^2 = 0$ and $\mathcal{W}$ has trivial cohomology. This reflects that fact that $\mathcal{W}$ serves as a model for the deRham complex of $E(G_1 \rtimes G_2)$, which is contractible by definition. In order to model $E(G_1 \rtimes G_2)$'s free $G_1 \rtimes G_2$ action, we require a degree -1 differential operator I_A whose action is defined as

$$I_A\theta^B = \delta_A{}^B \ , \qquad I_A\phi^B = 0 \ . \tag{B.12}$$

We can further define a degree zero differential operators L_A which encode the infinitesimal action of $G_1 \rtimes G_2$ on $(\mathfrak{g}_1 \rtimes \mathfrak{g}_2)^\vee$ through the co-adjoint representation. Here, we define it through $L_A = I_A d_{\mathcal{W}} + d_{\mathcal{W}} I_A$, giving

$$L_A\theta^B = -f_{AC}{}^B\theta^C \ , \qquad L_A\phi^B = -f_{AC}{}^B\phi^C \ , \tag{B.13}$$

or, in the split basis, as

$$L_a\theta^b = -f_{ac}{}^b\theta^c - f_{ak}{}^b\theta^k \ , \qquad L_a\phi^b = -f_{ac}{}^b\phi^c - f_{ak}{}^b\phi^k \ , \tag{B.14}$$

$$L_a\theta^i = 0 \ , \qquad L_a\phi^i = 0 \ , \tag{B.15}$$

and

$$L_i\theta^a = -f_{ic}{}^a\theta^c \ , \qquad\qquad L_i\phi^b = -f_{ic}{}^a\phi^c \ , \tag{B.16}$$

$$L_i\theta^j = -f_{ik}{}^j\theta^k \ , \qquad\qquad L_i\phi^j = -f_{ik}{}^j\phi^k \ . \tag{B.17}$$

Above, the mixed structure constants in $L_a\theta^a$ and $L_a\phi^b$ are crucial to our discussion as they encode the difference between a semi-direct product and a direct product action. When we come to observables, we will see how these terms spoil the invariant polynomials of $\mathfrak{g}_1$, namely preventing them from being basic classes.

C Ex. $\mathbb{X} = \mathbb{T}^4$ and $G = \mathbf{SU}(2)$

Here we work through the explicit example of the Weil algebra in this paper for the case $\mathbb{X}$ a four torus and $G = \mathrm{SU}(2)$.

C.1 Diffeomorphisms

Let $\mathbb{X} = \mathbb{T}^4 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ be the four torus (with the standard smooth structure). We are interested in the space of square integrable functions $\mathbb{X}$, namely the space $L^2(\mathbb{X})$. We have a basis for this space given by

$$f_{\vec{n}}(\vec{\theta}) = e^{i\vec{n}\cdot\vec{\theta}} \tag{C.1}$$

where $\vec{n} \in \mathbb{Z}^4$ is a four vector of integers and $\vec{\theta} \in \mathbb{X}$ are points on the four torus. Any function on X can then be written as a linear combination of the f_ns, i.e. for any function $F : \mathbb{X} \to \mathbb{C}$, we have

$$F(\vec{\theta}) = \sum_{\vec{n}\in\mathbb{Z}^4} F_{\vec{n}} f_{\vec{n}}(\vec{\theta}), \tag{C.2}$$

with $F_{\vec{n}} \in \mathbb{C}$.

Next, we wish to consider the space of orientation preserving diffeomorphism on $\mathbb{X}$, denoted by $\mathsf{Diff}_+(\mathbb{X})$. This space has a group structure given by the composition of maps. Locally, the Lie algebra of $\mathsf{Diff}_+(\mathbb{X})$, denoted by $\mathsf{diff}(\mathbb{X})$, is the space of all vector fields on $\mathbb{X}$ with Lie bracket given by the Lie derivative of vector fields. Explicitly, a basis for $\mathsf{diff}(\mathbb{X})$ is given by elements of the form

$$\eta_{(\vec{n},\mu)}(\vec{\theta}) = f_{\vec{n}}(\vec{\theta})\frac{\partial}{\partial\theta^\mu} = e^{i\vec{n}\cdot\vec{\theta}}\frac{\partial}{\partial\theta^\mu}. \tag{C.3}$$

We can then write any diffeomorphism η as

$$\eta(\vec{\theta}) = \sum_{\sigma=1}^{4} \sum_{\vec{n}\in\mathbb{Z}^4} \eta^{\sigma}_{\vec{n}} \eta_{(\vec{n},\sigma)}(\vec{\theta}) \tag{C.4}$$

with $\eta^{\sigma}_{\vec{n}} \in \mathbb{C}$. At times we will have occasion to write

$$\eta^{\sigma}(\vec{\theta}) = \sum_{n\in\mathbb{Z}^4} \eta^{\sigma}_{\vec{n}} e^{i\vec{n}\cdot\vec{\theta}}, \tag{C.5}$$

where we have conducted the sum over the Fourier modes. Employing summation rules, we can then write $\eta = \eta^{\sigma}\partial_{\sigma}$. The Lie bracket of $\mathsf{diff}(\mathbb{X})$ is

$$[\eta_{(\vec{n},\mu)}, \eta_{(\vec{m},\nu)}] = ie^{i(\vec{n}+\vec{m})\cdot\vec{\theta}} \left(m_{\mu}\frac{\partial}{\partial\theta^{\nu}} - n_{\nu}\frac{\partial}{\partial\theta^{\mu}} \right) = \sum_{(\vec{\ell},\sigma)} f_{(\vec{n},\mu)(\vec{m},\nu)}{}^{(\vec{\ell},\sigma)} \eta_{(\vec{\ell},\sigma)}. \tag{C.6}$$

Thus we can read off the structure constants as

$$f_{(\vec{n},\mu)(\vec{m},\nu)}{}^{(\vec{\ell},\sigma)} = i\delta^{\vec{\ell}}_{\vec{n}+\vec{m}} \left(m_{\mu}\delta^{\sigma}{}_{\nu} - n_{\nu}\delta^{\sigma}{}_{\mu} \right), \tag{C.7}$$

which are antisymmetric in the lowered indices, as expected.

C.2 Gauge transformation

Let us turn to gauge transformations. Take our gauge group as $G = \mathrm{SU}(2)$, with a basis of its Lie algebra $\mathfrak{su}(2)$, as

$$T_a = -\frac{i}{2}\tau_a, \tag{C.8}$$

where τ^a are the usual Pauli matrices. With this normalization of our basis, we have the relations

$$[T_a, T_b] = \epsilon_{ab}{}^{c} T_c, \tag{C.9}$$

where ${\epsilon_{ab}}^c$ is the usual antisymmetric tensor with ${\epsilon_{12}}^3 = +1$.

Now, consider the principal G bundle $P \to \mathbb{X}$. We define the group of gauge transformations $\mathcal{G}$ as space of fibre preserving automorphisms of P. Locally, we have Lie $\mathcal{G}$ as the space of maps from $\mathbb{X}$ to usual Lie algebra of $\mathfrak{su}(2)$. Thus, using the basis for $L^2(\mathbb{X})$ from the last subsection, we have a basis for Lie $\mathcal{G}$ given by elements of the form

$$T_{(\vec{n},a)}(\vec{\theta}) = f_{\vec{n}}(\vec{\theta})T_a = e^{i\vec{n}\cdot\vec{\theta}}T_a. \tag{C.10}$$

Thus we can write any gauge transformation ϵ as

$$\epsilon(\vec{\theta}) = \sum_{a=1}^{3}\sum_{\vec{n}\in\mathbb{Z}^4} \epsilon^a_{\vec{n}} T_{(\vec{n},a)}(\vec{\theta}), \tag{C.11}$$

with $\epsilon^a_{\vec{n}} \in \mathbb{C}$. As in the case of diffeomorphisms, we will on occasion will conduct the sum over Fourier modes and write ϵ^a, so that $\epsilon = \epsilon^a T_a$, where, again, we implicitly sum over a. The structure constants for the Lie algebra of gauge transformations follow directly from those of the Lie algebra of the gauge group including the effect of the $f_{\vec{n}}$s. We have

$${f_{(\vec{n},a)(\vec{m},b)}}^{(\vec{\ell},c)} = {\epsilon_{ab}}^c \delta^{\vec{\ell}}_{\vec{n}+\vec{m}}. \tag{C.12}$$

C.3 Diffeomorphisms & Gauge transformations

Next, we want to take the semi-direct product of $\mathsf{Diff}_+(\mathbb{X})$ and $\mathcal{G}$ acting on the space of adjoint valued differential forms on $\mathbb{X}$. To see that we want a semi-direct product, take $g \in \mathcal{G}$ and $f \in \mathsf{Diff}_+(\mathbb{X})$ and consider their actions on some $\phi \in \Omega^0(\mathbb{X}, \mathrm{ad}\, P)$. We have, by definition,

$$(g \circ \phi)(x) = g(x)\phi(x)g(x)^{-1} \quad \text{and} \quad (f \circ \phi)(x) = \phi(f(x)). \tag{C.13}$$

Thus

$$(f\circ(g\circ\phi))(x) = f\circ(g(x)\phi(x)g(x)^{-1}) = g(f(x))\phi(f(x))g(f(x))^{-1} \tag{C.14}$$

and

$$(g\circ(f\circ\phi))(x) = g\circ(\phi(f(x))) = g(x)\phi(f(x))g(x)^{-1}, \tag{C.15}$$

which together give

$$f\circ(g\circ f^{-1}) = f^*(g). \tag{C.16}$$

This tells us that we do indeed wish to consider $\mathbb{G} = \mathcal{G}\rtimes \mathsf{Diff}_+(\mathbb{X})$. In the local Lie algebra description, this semi-direct product is specified by a Lie algebra homomorphism $\rho : \mathsf{diff}(\mathbb{X}) \to \mathrm{Der}(\mathrm{Lie}\,\mathcal{G})$, where the target is the space of derivations of $\mathrm{Lie}\,\mathcal{G}$. Given $(\epsilon_1,\eta_1),(\epsilon_2,\eta_2)\in \mathrm{Lie}\,\mathcal{G}\oplus\mathsf{diff}(\mathbb{X})$, our Lie bracket is given by

$$[(\epsilon_1,\eta_1),(\epsilon_2,\eta_2)] = ([\epsilon_1,\epsilon_2]+\rho(\eta_1)(\epsilon_2)-\rho(\eta_2)(\epsilon_1),[\eta_1,\eta_2]). \tag{C.17}$$

The particular homomorphism we are interested in is simply $\rho(\eta)=\eta^\sigma\partial_\sigma$, where η acts as a vector field on gauge transformations. We can then write

$$[[(\epsilon_1,\eta_1),(\epsilon_2,\eta_2)][= ([\epsilon_1,\epsilon_2]+\eta_1^\sigma\partial_\sigma\epsilon_2-\eta_2^\sigma\partial_\sigma\epsilon_1,[\eta_1,\eta_2]). \tag{C.18}$$

We would like to identify the structure constants of our Lie bracket. These will be the same as those of the previous two section along with additional contributions on the gauge transformation side due to the semi-direct product structure. One finds

$$[(T_{(\vec{n},a)},\eta_{(\vec{n}',\mu)}),(T_{(\vec{m},b)},\eta_{(\vec{m}',\nu)})] = \sum_{(\vec{\ell},c),(\vec{\ell}',\mu)}\Big(\epsilon_{ab}{}^c\delta^{\vec{\ell}}_{\vec{n}+\vec{m}}+im_\mu\delta^{\vec{\ell}}_{\vec{n}'+\vec{m}}\delta^c_b-in_\nu\delta^{\vec{\ell}}_{\vec{m}'+\vec{n}}\delta^c_a,$$

$$i\delta^{\vec{\ell}'}_{\vec{n}'+\vec{m}'}(m'_\mu\delta^\sigma{}_\nu - n'_\nu\delta^\sigma{}_\mu)\Big)\,(T_{(\vec{\ell},c)},\eta_{(\vec{\ell}',\sigma)}) \tag{C.19}$$

Or, with an unpalatable number of indices,

$$f_{\{(\vec{n},a),(\vec{n}',\mu)\},\{(\vec{m},b),(\vec{m}',\nu)\}}{}^{\{(\vec{\ell},c),(\vec{\ell}',\sigma)\}} = \Big(\epsilon_{ab}{}^c\delta^{\vec{\ell}}_{\vec{n}+\vec{m}} + im_\mu\delta^{\vec{\ell}}_{\vec{n}'+\vec{m}}\delta^c_b - in_\nu\delta^{\vec{\ell}}_{\vec{m}'+\vec{n}}\delta^c_a,$$
$$i\delta^{\vec{\ell}'}_{\vec{n}'+\vec{m}'}(m'_\mu\delta^\sigma{}_\nu - n'_\nu\delta^\sigma{}_\mu)\Big) \tag{C.20}$$

C.4 Weil Algebra

With an explicit understanding of $\mathrm{Lie}\,\mathbb{G}$, we now turn to constructing the Weil algebra. It is defined as

$$\mathcal{W}(\mathrm{Lie}\,\mathbb{G}) = S(\mathrm{Lie}\,\mathcal{G}\oplus\mathsf{diff}(\mathbb{X}))\otimes\Lambda(\mathrm{Lie}\,\mathcal{G}\oplus\mathsf{diff}(\mathbb{X})). \tag{C.21}$$

We can take generators of this algebra as the degree two elements of the symmetric algebra $\{\varphi^{\mathbb{A}}\}$ and the degree one elements of the exterior algebra $\{\theta^{\mathbb{A}}\}$. Here, the index $\mathbb{A}$ is of the form $((\vec{n},a),(\vec{n}',\mu))$, where the first tuple specifies a gauge transformation basis element and the second specifies a diffeomorphism basis element. In order to specify a complete set of generators we need not let $\mathbb{A}$ run over all such indices, and instead take the split basis. We define

$$\theta^{((\vec{n},a),(0,0))} = c^{(\vec{n},a)}\otimes 1 \qquad \varphi^{((\vec{n},a),(0,0))} = \phi^{(\vec{n},a)}\otimes 1 \tag{C.22}$$
$$\theta^{((0,0),(\vec{n},\mu))} = 1\otimes\xi^{(\vec{n},\mu)} \qquad \varphi^{((0,0),(\vec{n},\mu))} = 1\otimes\Phi^{(\vec{n},\mu)} \tag{C.23}$$

where $(0,0)$ indicates zero in the respective direct sum index. To avoid clutter, we make a redefinition of notation, so that now $\mathbb{A}$ runs over $(\vec{n}, A)$ with A running over all spatial indices $\mu = 1,2,3,4$ and gauge indices $a = 1,2,3$. Additionally, we will frequently suppress identity tensorands. With these indices, we can rewrite the structure constants of Lie $\mathbb{G}$ as

$$\begin{aligned} f_{\mathbb{A}\mathbb{B}}{}^{\mathbb{C}} &= f_{(\vec{n},A),(\vec{m},B)}{}^{(\vec{\ell},C)} \\ &= (\epsilon_{ab}{}^{c}\delta^a_A\delta^c_B\delta^C_c + im_\mu\delta^\mu_A\delta^b_B\delta^C_c\delta^c_b - in_\mu\delta^a_A\delta^\mu_B\delta^C_c\delta^c_a + im_\mu\delta^\mu_A\delta^\nu_B\delta^C_\sigma\delta^\sigma_\nu - in_\mu\delta^\nu_A\delta^\mu_B\delta^C_\sigma\delta^\sigma_\nu)\delta^{\vec{\ell}}_{\vec{n}+\vec{m}} \end{aligned} \tag{C.24}$$

Our Weil differential act on the generators as

$$d_{\mathcal{W}}\theta^{\mathbb{A}} = \varphi^{\mathbb{A}} - \frac{1}{2}[[\theta,\theta]]^{\mathbb{A}}, \qquad d_{\mathcal{W}}\varphi^{\mathbb{A}} = -[[\theta,\varphi]]^{\mathbb{A}}. \tag{C.25}$$

Further, we have the degree -1 differential operator $I_{\mathbb{A}}$ as

$$I_{\mathbb{A}}\theta^{\mathbb{B}} = \delta^{\mathbb{B}}_{\mathbb{A}} \qquad I_{\mathbb{A}}\varphi^{\mathbb{B}} = 0. \tag{C.26}$$

and the degree zero Lie derivative $L_{\mathbb{A}}$ as

$$L_{\mathbb{A}} = I_{\mathbb{A}}d_{\mathcal{W}} + d_{\mathcal{W}}I_{\mathbb{A}}. \tag{C.27}$$

We wish to identify the action of $d_{\mathcal{W}}$, $I_{\mathbb{A}}$, and $L_{\mathbb{A}}$ in this example. Starting with the Weil differential, we have

$$d_{\mathcal{W}}\xi^{(\vec{n},\mu)} = \varphi^{(\vec{n},\mu)} - \frac{1}{2}[[\theta,\theta]]^{(\vec{n},\mu)} = \Phi^{(\vec{n},\mu)} - \sum_{\mathbb{A}\mathbb{B}} f_{\mathbb{A}\mathbb{B}}{}^{(\vec{n},\mu)}\theta^{\mathbb{A}}\theta^{\mathbb{B}} \tag{C.28}$$

$$= \Phi^{(\vec{n},\mu)} - \frac{1}{2} i \sum_{\vec{m}\vec{\ell}} \sum_{\nu\sigma} \delta^{\vec{n}}_{\vec{m}+\vec{\ell}} (\ell_\nu \delta^\mu_\sigma - m_\sigma \delta^\mu_\nu) \theta^{(\vec{m},\nu)} \theta^{(\vec{\ell},\sigma)} \tag{C.29}$$

$$= \Phi^{(\vec{n},\mu)} - \frac{i}{2} \sum_{\vec{m}\sigma} ((\vec{n}-\vec{m})_\sigma \xi^{(\vec{p},\sigma)} \xi^{(\vec{n}-\vec{m},\mu)} - \vec{m}_\sigma \xi^{(\vec{m},\mu)} \xi^{(\vec{n}-\vec{m},\sigma)}) \tag{C.30}$$

$$= \Phi^{(\vec{n},\mu)} - \sum_{\vec{m}\sigma} \xi^{(\vec{m},\sigma)} i (\vec{n}-\vec{p})_\sigma \xi^{(\vec{n}-\vec{m},\mu)}, \tag{C.31}$$

where in the last line we have resummed the second term in the penultimate expression and used the fact that the ξs anticommute. We can also recognize in the above expression the fact that $i(\vec{n}-\vec{m})_\sigma$ is value of ∂_σ on $f_{\vec{n}-\vec{m}}$. Therefore, we sum over $\vec{n}$ on both sides and write

$$d_{\mathcal{W}} \xi^\mu = \Phi^\mu - \xi^\sigma \partial_\sigma \xi^\mu. \tag{C.32}$$

Similarily, we have

$$d_{\mathcal{W}} \Phi^{(\vec{n},\mu)} = -[[\theta, \varphi]]^{(\vec{n},\mu)} \tag{C.33}$$

$$= -\sum_{\vec{m}\sigma} (i(\vec{n}-\vec{m})_\sigma \xi^{(\vec{m},\sigma)} \Phi^{(\vec{n}-\vec{m},\mu)} - i\vec{m}_\sigma \xi^{(\vec{m},\mu)} \Phi^{(\vec{n}-\vec{m},\sigma)}), \tag{C.34}$$

and again we can sum over n on both sides to write

$$d_{\mathcal{W}} \Phi^\mu = -\xi^\sigma \partial_\sigma \Phi^\mu + \Phi^\sigma \partial_\sigma \xi^\mu. \tag{C.35}$$

Turning to c and ϕ, we must be careful with the extra factors in the structure constants. We find

$$d_{\mathcal{W}} c^{(\vec{n},a)} = \varphi^{(\vec{n},a)} - \frac{1}{2}[\theta, \theta]^{(\vec{n},a)} = \phi^{(\vec{n},a)} - \frac{1}{2} \sum_{\mathbb{A}\mathbb{B}} f_{\mathbb{A}\mathbb{B}}{}^{(\vec{n},a)} \theta^{\mathbb{A}} \theta^{\mathbb{B}} \tag{C.36}$$

$$= \phi^{(\vec{n},a)} - \frac{1}{2} \sum_{\vec{m}bc} \epsilon_{bc}{}^a c^{(\vec{m},b)} c^{(\vec{n}-\vec{m},c)} - \frac{i}{2} \sum_{\vec{m}\sigma} m_\nu \xi^{(\vec{n}-\vec{m},\sigma)} c^{(\vec{m},a)} + \frac{i}{2} \sum_{\vec{m}\sigma} m_\sigma c^{(\vec{m},a)} \xi^{(\vec{n}-\vec{m},\sigma)} \tag{C.37}$$

$$= \phi^{(\vec{n},a)} - \frac{1}{2}\sum_{\vec{m}bc} \epsilon_{bc}{}^{a} c^{(\vec{m},b)} c^{(\vec{n}-\vec{m},c)} - \sum_{\vec{m}\sigma} \xi^{(\vec{n}-\vec{m},\sigma)} i m_\sigma c^{(\vec{m},a)}. \tag{C.38}$$

Here we can sum both sides over n and rewrite the result as

$$d_{\mathcal{W}} c^a = \phi^a - \frac{1}{2}[c,c]^a - \xi^\sigma \partial_\sigma c^a. \tag{C.39}$$

Next, for ϕ we have

$$d_{\mathcal{W}} \phi^{(\vec{n},a)} = -[\theta, \varphi]^{(\vec{n},a)} \tag{C.40}$$

$$= -\sum_{\vec{p}bc} \epsilon_{bc}{}^{a} c^{(\vec{m},b)} \phi^{(\vec{n}-\vec{m},c)} - i\sum_{\vec{m}\sigma} m_\sigma \xi^{(\vec{n}-\vec{m},\sigma)} \phi^{(\vec{m},a)} + i\sum_{\vec{m}\sigma} m_\sigma c^{(\vec{m},a)} \Phi^{(\vec{n}-\vec{m},\sigma)}, \tag{C.41}$$

where again we sum over n on both sides to obtain

$$d_{\mathcal{W}} \phi^a = -[c,\phi]^a - \xi^\sigma \partial_\sigma \phi^a + \Phi^\sigma \partial_\sigma c. \tag{C.42}$$

Next, the interior derivatives are easily computed as

$$I_{(\vec{n},a)} c^{(\vec{m},b)} = \delta^{\vec{m}}_{\vec{n}} \delta^b_a, \qquad I_{(\vec{n}',\mu)} \xi^{(\vec{m},\nu)} = \delta^{\vec{m}}_{\vec{n}} \delta^\mu_\mu, \tag{C.43}$$

$$I_{(\vec{n},a)} \phi^{(\vec{m},b)} = 0, \qquad I_{(\vec{n}',\mu)} \Phi^{(\vec{m},\nu)} = 0. \tag{C.44}$$

Finally we turn to the Lie derivative. Since $I_{\mathbb{A}}$ yields either zero or delta functions, and $d_{\mathcal{W}}$ vanishes on delta functions, we only need to compute $I_{\mathbb{A}} d_{\mathcal{W}}$ on our generators to understand the action of $L_{\mathbb{A}}$. We have for the gauge indexed operators

$$L_{(\vec{n},a)} c^{(\vec{m},b)} = -\sum_c \epsilon_{ac}{}^{b} c^{(\vec{m}-\vec{n},c)} + \sum_\sigma \xi^{(\vec{m}-\vec{n},\sigma)} i n_\nu \delta^b_a, \qquad L_{(\vec{n},a)} \xi^{(\vec{m},\sigma)} = 0, \tag{C.45}$$

$$L_{(\vec{n},a)}\phi^{(\vec{m},b)} = -\sum_c \epsilon_{ac}{}^b \phi^{(\vec{m}-\vec{n},c)} + \sum_\sigma i n_\sigma \Phi^{(\vec{m}-\vec{n},\sigma)}\delta^b_a, \qquad L_{(\vec{n},a)}\Phi^{(\vec{m},\nu)} = 0, \tag{C.46}$$

and for the diffeomorphism indexed operators,

$$L_{(\vec{n},\mu)}c^{(\vec{m},b)} = -i(\vec{m}-\vec{n})_\mu c^{(\vec{m}-\vec{n},b)}, \quad L_{(\vec{n},\mu)}\xi^{(\vec{m},\nu)} = -i(\vec{m}-\vec{n})_\mu \xi^{(\vec{m}-\vec{n},\nu)} + \sum_\sigma \xi^{(\vec{m}-\vec{n},\sigma)} i n_\sigma \delta^\nu_\mu, \tag{C.47}$$

$$L_{(\vec{n},\mu)}\phi^{(\vec{m},b)} = -i(\vec{m}-\vec{n})_\mu \phi^{(\vec{m}-\vec{n},b)}, \quad L_{(\vec{n},\mu)}\Phi^{(\vec{m},\nu)} = -i(\vec{m}-\vec{n})_\mu \Phi^{(\vec{m}-\vec{n},\nu)} + \sum_\sigma \Phi^{(\vec{m}-\vec{n},\sigma)} i n_\sigma \delta^\nu_\mu. \tag{C.48}$$

In order to return this to a more palatable form, let us contract each Lie derivative by the associated degree one generator, and then conduct the sums over both $\vec{n}$ and $\vec{m}$. Allowing ourselves an abusive, but, at this point, well understood notation, we have

$$L_c c^b := c^a(L_a c^b) = -[c,c]^b - \xi^\sigma \partial_\sigma c^b, \qquad L_c \xi^\mu := c^a(L_a \xi^\mu) = 0, \tag{C.49}$$

$$L_c \phi^b := c^a(L_a \phi^b) = -[c,\phi]^b + \Phi^\sigma \partial_\sigma c^b, \qquad L_c \Phi^\mu := c^a(L_a \Phi^\mu) = 0, \tag{C.50}$$

and

$$L_\xi c^b := \xi^\sigma(L_\sigma c^b) = -\xi^\sigma \partial_\sigma c^b, \qquad L_\xi \xi^\mu := \xi^\sigma(L_\sigma \xi^\mu) = -2\xi^\sigma \partial_\sigma \xi^\mu, \tag{C.51}$$

$$L_\xi \phi^b := \xi^\sigma(L_\sigma \phi^b) = -\xi^\sigma \partial_\sigma \phi^b, \quad L_\xi \Phi^\mu := \xi^\sigma(L_\sigma \Phi^\mu) = -\xi^\sigma \partial_\sigma \Phi^\mu + \Phi^\sigma \partial_\sigma \xi^\mu. \tag{C.52}$$

Note that

$$\theta^{\mathbb{A}} L_{\mathbb{A}} \varphi^{\mathbb{B}} = d_{\mathcal{W}} \varphi^{\mathbb{B}}. \tag{C.53}$$

D Variation of Self-Dual Fields

Suppose that we have a (anti-) self-dual field $\omega \in \Omega^{2,\pm}(\mathbb{X})$, such that

$$\omega = \pm \star \omega, \tag{D.1}$$

where here and throughout the section the top sign is for self-dual and the lower for anti self-dual ω. Under a change in the metric $g \to g + \delta g$, we have $\omega \to \omega + \delta_g \omega$. If we want to maintain the duality condition for the new perturbed metric, we require

$$\delta_g \omega = \pm \delta_g (\star \omega) = \pm(\delta_g \star)\omega \pm \star(\delta_g \omega), \tag{D.2}$$

or, equivalently, that

$$\frac{1}{2}(1 \mp \star)\delta_g \omega = \pm \frac{1}{2}(\delta_g \star)\omega. \tag{D.3}$$

Hence, the variation of a (anti-) self-dual field will have a contribution from the variation of the Hodge star. This contribution is the opposite duality of the field with respect to the unperturbed metric.

To find an explicit coordinate expression for this contribution, let us turn to the case at hand, where $\delta g_{\mu\nu} = \Psi_{\mu\nu}$. Our conventions take condition (D.1) as

$$\omega_{\mu\nu} = \pm \frac{1}{2}\sqrt{g}\epsilon_{\mu\nu\rho\sigma} g^{\rho\rho'} g^{\sigma\sigma'} \omega_{\rho'\sigma'}, \tag{D.4}$$

where $\sqrt{g} = \sqrt{\det\ g_{\mu\nu}}$. Our Levi-Civita symbol has no metric dependence, and

$$g g^{\mu\mu'} g^{\nu\nu'} g^{\rho\rho'} g^{\sigma\sigma'} \epsilon_{\mu'\nu'\rho'\sigma'} = \epsilon^{\mu\nu\rho\sigma}, \tag{D.5}$$

Further, we will often use

$$\epsilon^{\delta\rho\sigma\tau}\epsilon_{\mu\nu\gamma\tau} = \delta^{\delta}_{\mu}(\delta^{\rho}_{\nu}\delta^{\sigma}_{\gamma} - \delta^{\sigma}_{\nu}\delta^{\rho}_{\gamma}) - \delta^{\rho}_{\mu}(\delta^{\delta}_{\nu}\delta^{\sigma}_{\gamma} - \delta^{\sigma}_{\nu}\delta^{\delta}_{\gamma}) + \delta^{\sigma}_{\mu}(\delta^{\delta}_{\nu}\delta^{\rho}_{\gamma} - \delta^{\rho}_{\nu}\delta^{\delta}_{\gamma}). \tag{D.6}$$

Conventions settled, we need to lift the differential of $H_{\mathsf{Diff}_+(\mathbb{X})}(\mathsf{Met}(\mathbb{X}))$ to the total space of the projected bundle $\Omega^{2,\pm}_g(\mathbb{X})$ over $\mathsf{Met}(\mathbb{X})$. Since the projection depends continuous on the metric, this bundle is non-trivial, unlike the unprojected bundle $\Omega^2(\mathbb{X})$. Therefore, the lift of d, denoted $\widetilde{\mathsf{d}}$ will be a projection connection.[46] We have

$$(\mathsf{d}\star)\omega = \frac{1}{2}\Psi^{\sigma}{}_{\sigma}\omega_{\mu\nu} \mp \sqrt{g}\epsilon_{\mu\nu\rho\sigma}\Psi^{\rho\gamma}\omega_{\gamma}{}^{\sigma}. \tag{D.7}$$

Note that

$$\Psi^{\sigma}{}_{[\mu}\omega_{\nu]\sigma} = -\frac{1}{2}\Psi^{\sigma}{}_{\sigma}\omega_{\mu\nu} \pm \frac{1}{2}\sqrt{g}\epsilon_{\mu\nu\rho\gamma}\Psi^{\rho\sigma}\omega_{\sigma}{}^{\gamma}, \tag{D.8}$$

which follows from the identity

$$\sqrt{g}\epsilon_{\mu\nu\rho\gamma}A^{(\rho\sigma)}B_{\sigma}{}^{\gamma} = \pm A^{\sigma}{}_{\sigma}B_{\mu\nu} \pm 2A^{\sigma}{}_{[\mu}B_{\nu]\sigma}, \tag{D.9}$$

for any symmetric two-tensor A and (anti-) self-dual two-form B. In particular, we find that

$$(\Psi^{\sigma}{}_{[\mu}\omega_{\nu]\sigma})^{\mp} = -\frac{1}{4}\Psi^{\sigma}{}_{\sigma}\omega_{\mu\nu} \pm \frac{1}{2}\sqrt{g}\epsilon_{\mu\nu\rho\gamma}\Psi^{\rho}{}_{\sigma}\omega^{\sigma\gamma}, \tag{D.10}$$

$$(\Psi^{\sigma}{}_{[\mu}\omega_{\nu]\sigma})^{\pm} = -\frac{1}{4}\Psi^{\sigma}{}_{\sigma}\omega_{\mu\nu}, \tag{D.11}$$

where the raised signs indicate explicit projection to self-dual or anti-dual parts via the projector $\frac{1}{2}(1 \pm \star)$. Hence, we can satisfy (D.3) by adding $-(\Psi^{\sigma}{}_{[\mu}\omega_{\nu]\sigma})^{\mp}$ to the

[46] For more on the importance of projected connections in physics, do not overlook [69]

variation of ω. All together, we arrive at we have

$$\widetilde{\mathsf{d}}\omega_{\mu\nu} = (\mathsf{d}\omega_{\mu\nu})^{\pm} - (\Psi^{\sigma}{}_{[\mu}\omega_{\nu]\sigma})^{\mp}, \tag{D.12}$$

where we have denoted the lift of d to the total space of the bundle $\Omega^2(\mathbb{X})$ over $\mathsf{Met}\mathbb{X}$ with the same symbol, as it is canonical. In our conventions, we assume that all unconstrained differential forms have no metric dependence, and thus $\mathsf{d}\omega = 0$, and we end up with

$$\widetilde{\mathsf{d}}\omega_{\mu\nu} = -(\Psi^{\sigma}{}_{[\mu}\omega_{\nu]\sigma})^{\mp}. \tag{D.13}$$

The story is different if we are varying the (anti-) self-dual part of an otherwise unconstrained two-form field. For example, we will encounter the variation of F_A^+. Since F_A has both self-dual and anti-self-dual components, we can't immediately leap to (D.13). Instead, in complete generality, suppose we have an unconstrained two-form field $\varpi \in \Omega^2(\mathbb{X})$. Then

$$\begin{aligned}
\mathsf{d}(\varpi^{\pm}_{\mu\nu}) &= \mathsf{d}\left(\frac{1}{2}\varpi_{\mu\nu} \pm \frac{1}{4}\sqrt{g}\epsilon_{\mu\nu\rho\sigma}\varpi^{\rho\sigma}\right), \\
&= \frac{1}{2}(\mathsf{d}\varpi_{\mu\nu}) \pm \frac{1}{4}\sqrt{g}\epsilon_{\mu\nu\rho\sigma}g^{\rho\rho'}g^{\sigma\sigma'}\mathsf{d}\varpi_{\rho'\sigma'} \\
&\qquad \pm \frac{1}{2}\sqrt{g}\epsilon_{\mu\nu\rho\sigma}\left(\frac{1}{4}\Psi^{\gamma}{}_{\gamma}g^{\rho\rho'}g^{\sigma\sigma'} - \Psi^{\rho\rho'}g^{\sigma\sigma'}\right)\varpi_{\rho'\sigma'} \\
&= (\mathsf{d}\varpi)^{\pm}_{\mu\nu} \pm \frac{1}{2}\sqrt{g}\epsilon_{\mu\nu\rho\sigma}\left(\frac{1}{4}\Psi^{\gamma}{}_{\gamma}(\varpi^{\rho\sigma,+} + \varpi^{\rho\sigma,-}) - \Psi^{\rho}{}_{\gamma}(\varpi^{\gamma\sigma,+} + \varpi^{\gamma\sigma,-})\right) \\
&= \mp(\Psi^{\sigma}{}_{[\mu}\varpi^{+}_{\nu]\sigma})^{-} \pm (\Psi^{\sigma}{}_{[\mu}\varpi^{-}_{\nu]\sigma})^{+},
\end{aligned} \tag{D.14}$$

where we have again used $\mathsf{d}\varpi = 0$. Note that if $\varpi^{\mp} = 0$, this reproduces (D.13).

Turning to the square $\widetilde{\mathsf{d}}^2$ on $\omega \in \Omega^{2,\pm}_g(\mathbb{X})$ and d^2 on $\varpi \in \Omega^2(\mathbb{X})$, there is an incredibly important difference. Specializing to the self-dual projection of interest, we can compute

$$\begin{aligned}
\mathsf{d}^2\varpi^+_{\mu\nu} &= \frac{1}{4}\sqrt{g}\epsilon_{\mu\nu\rho\sigma}\Psi^\gamma{}_\gamma\left(\frac{1}{4}\Psi^\tau{}_\tau g^{\rho\rho'}g^{\sigma\sigma'} - \Psi^{\rho\rho'}g^{\sigma\sigma'}\right)\omega_{\rho'\sigma'} \\
&\quad + \frac{1}{2}\sqrt{g}\epsilon_{\mu\nu\rho\sigma}\left(\frac{1}{2}(\nabla^\gamma\Phi_\gamma)g^{\rho\rho'}g^{\sigma\sigma'} - (\nabla^\rho\Phi^{\rho'} + \nabla^{\rho'}\Phi^\rho)g^{\sigma\sigma'}\right)\varpi_{\rho'\sigma'} \\
&\quad + \frac{1}{2}\sqrt{g}\epsilon_{\mu\nu\rho\sigma}\left(\frac{1}{2}\Psi^\gamma{}_\gamma\Psi^{\rho\rho'}g^{\sigma\sigma'} + \Psi^{\rho\rho'}\Psi^{\sigma\sigma'}\right)\varpi_{\rho'\sigma'} \\
&= \frac{1}{2}\sqrt{g}\epsilon_{\mu\nu\rho\sigma}\left(\frac{1}{2}(\nabla_\gamma\Phi^\gamma)g^{\rho\rho'}g^{\sigma\sigma'} - (\nabla^\rho\Phi^{\rho'} + \nabla^{\rho'}\Phi^\rho)g^{\sigma\sigma'}\right)\varpi_{\rho'\sigma'}.
\end{aligned} \tag{D.15}$$

This coincides with the expectation that $\mathsf{d}^2 = \mathcal{L}_\Phi$, as the final line is the Lie derivative of the Hodge star operator. On the other hand, we have

$$\begin{aligned}
\widetilde{\mathsf{d}}^2\omega_{\mu\nu} &= \frac{1}{2}\sqrt{g}\epsilon_{\mu\nu\rho\sigma}\left(\frac{1}{2}(\nabla_\gamma\Phi^\gamma)g^{\rho\rho'}g^{\sigma\sigma'} - (\nabla^\rho\Phi^{\rho'} + \nabla^{\rho'}\Phi^\rho)g^{\sigma\sigma'}\right)\omega_{\rho'\sigma'} \\
&\quad - \frac{1}{2}\sqrt{g}\epsilon_{\mu\nu\rho\sigma}\left(\frac{1}{4}\Psi^\gamma{}_\gamma g^{\rho\rho'}g^{\sigma\sigma'} - \Psi^{\rho\rho'}g^{\sigma\sigma'}\right)\widetilde{\mathsf{d}}\omega_{\rho'\sigma'}.
\end{aligned} \tag{D.16}$$

Here, we encounter a new term in the second line, resulting from the fact that $\omega \in \Omega^{2,+}_g(\mathbb{X})$ as opposed to $\Omega^2(\mathbb{X})$. In a lengthy but important computation

$$\begin{aligned}
(\widetilde{\mathsf{d}}^2\varpi_{\mu\nu})_{\Psi\Psi} &= -\frac{1}{2}\sqrt{g}\epsilon_{\mu\nu\rho\sigma}\left(\frac{1}{4}\Psi^\gamma{}_\gamma g^{\rho\rho'}g^{\sigma\sigma'} - \Psi^{\rho\rho'}g^{\sigma\sigma'}\right)\widetilde{\mathsf{d}}\omega_{\rho'\sigma'} \\
&= -\frac{1}{2}\sqrt{g}\epsilon_{\mu\nu\rho\sigma}\left(\frac{1}{4}\Psi^\gamma{}_\gamma g^{\rho\rho'}g^{\sigma\sigma'} - \Psi^{\rho\rho'}g^{\sigma\sigma'}\right) \\
&\qquad \times\left(\frac{1}{2}\sqrt{g}\epsilon_{\rho'\sigma'\lambda\tau}\left(\frac{1}{4}\Psi^\eta{}_\eta g^{\lambda\lambda'}g^{\tau\tau'} - \Psi^{\lambda\lambda'}g^{\tau\tau'}\right)\varpi_{\lambda'\lambda'}\right) \\
&= -\frac{1}{16}g\epsilon_{\mu\nu\rho\sigma}\epsilon_{\rho'\sigma'\lambda\tau}\left(-g^{\rho\rho'}g^{\sigma\sigma'}g^{\tau\tau'}\Psi^\gamma{}_\gamma\Psi^{\lambda\lambda'} + g^{\lambda\lambda'}g^{\sigma\sigma'}g^{\tau\tau'}\Psi^\eta{}_\eta\psi^{\rho\rho'}\right)\varpi_{\lambda'\tau'} \\
&\qquad - \frac{1}{4}g\epsilon_{\mu\nu\rho\sigma}\epsilon_{\rho'\sigma'\lambda\tau}g^{\sigma\sigma'}g^{\tau\tau'}\Psi^{\rho\rho'}\Psi^{\lambda\lambda'}\varpi_{\lambda'\tau'} \\
&= \frac{1}{16}\epsilon_{\mu\nu\rho\sigma}\left(\epsilon^{\rho\sigma\lambda\tau}\Psi^\gamma{}_\gamma\Psi_\lambda{}^\eta\varpi_{\eta y} - \epsilon^{\eta\sigma\lambda\tau}\Psi^\gamma{}_\gamma\Psi^\rho{}_\eta\varpi_{\lambda\tau} - 4\epsilon^{\eta\sigma\lambda\tau}\Psi^\rho{}_\eta\Psi_\lambda{}^\gamma\varpi_{\gamma\tau}\right) \\
&= -\frac{1}{16}(\delta^\rho_\mu(\delta^\lambda_\nu\delta^\tau_\eta - \delta^\tau_\nu\delta^\lambda_\eta) - \delta^\lambda_\mu(\delta^\rho_\nu\delta^\tau_\eta - \delta^\tau_\nu\delta^\rho_\eta) + \delta^\tau_\mu(\delta^\rho_\nu\delta^\lambda_\eta - \delta^\lambda_\nu\delta^\rho_\eta))
\end{aligned} \tag{D.17}$$

$$
\begin{aligned}
&\quad \times (\Psi^\sigma{}_\sigma \Psi^\eta{}_\rho \varpi_{\lambda\tau} + 4\Psi^\eta{}_\rho \Psi_\lambda{}^\sigma \varpi_{\sigma\tau}) + \frac{1}{8}(\delta^\lambda_\mu \delta^\tau_\nu - \delta^\tau_\mu \delta^\lambda_\nu)\Psi^\sigma{}_\sigma \Psi_\lambda{}^\rho \varpi_{\rho\tau} \\
&= -\frac{1}{4}(\Psi^\rho{}_\mu \Psi_\nu{}^\sigma \varpi_{\sigma\rho} - \Psi^\rho{}_\mu \Psi_\rho{}^\sigma \varpi_{\sigma\nu} - \Psi^\rho{}_\nu \Psi_\mu{}^\sigma \varpi_{\sigma\rho} \\
&\quad + \Psi^\rho{}_\rho \Psi_\mu{}^\sigma \varpi_{\sigma\nu} + \Psi^\rho{}_\nu \Psi_\rho{}^\sigma \varpi_{\sigma\mu} - \Psi^\rho{}_\rho \Psi_\nu{}^\sigma \varpi_{\sigma\mu}) \\
&\quad - \frac{1}{4}\Psi^\sigma{}_\sigma \Psi^\rho{}_{[\mu} \varpi_{\nu]\rho} - \frac{1}{8}\Psi^\sigma{}_\sigma \Psi^\rho{}_\mu \varpi_{\nu\rho} + \frac{1}{8}\Psi^\sigma{}_\sigma \Psi^\rho{}_\nu \varpi_{\mu\rho} \\
&= -\frac{1}{2}\Psi^\sigma{}_\sigma \Psi^\rho{}_{[\mu} \varpi_{\nu]\rho} + \frac{1}{2}\Psi^\sigma{}_\sigma \Psi^\rho{}_{[\mu} \varpi_{\nu]\rho} + \frac{1}{2}\Psi^{\rho\sigma}\Psi_{\rho[\mu}\varpi_{\nu]\sigma} \\
&= \frac{1}{2}\Psi^{\rho\sigma}\Psi_{\rho[\mu}\varpi_{\nu]\sigma}. \qquad \text{(D.18)}
\end{aligned}
$$

All told, we find

$$
\widetilde{\mathsf{d}}^2 \omega_{\mu\nu} = \frac{1}{2}\sqrt{g}\epsilon_{\mu\nu\rho\sigma}\left(\frac{1}{2}(\nabla_\gamma \Phi^\gamma) g^{\rho\rho'} g^{\sigma\sigma'} - (\nabla^\rho \Phi^{\rho'} + \nabla^{\rho'}\Phi^\rho) g^{\sigma\sigma'}\right)\chi_{\rho'\sigma'} + \frac{1}{2}\Psi^{\rho\sigma}\Psi_{\rho[\mu}\chi_{\nu]\sigma}. \qquad \text{(D.19)}
$$

For our theory, the extra final term will be compensated by the Δ_H differential. We also recognize it as the curvature of the projection connection, and its existence leads us to conclude that the self-dual condition is inconsistent with the diffeomorphism Cartan model alone and requires the combined gauge and diffeomorphism model. In other words, modules with self-dual fields cannot be constructed for $H_{\mathsf{Diff}_+(\mathbb{X})}(\mathsf{Met}(\mathbb{X}))$ but can be for $H_{\mathbb{G}}(\mathbb{M})$.

It is instructive to see how this leads to the correct algebra for our Cartan model. We have the very important

$$
\mathbb{Q}\varpi^\pm_{\mu\nu} = (\mathbb{Q}\varpi)^\pm_{\mu\nu} \mp (\Psi^\sigma{}_{[\mu}\varpi^+_{\nu]\sigma})^- \pm (\Psi^\sigma{}_{[\mu}\varpi^-_{\nu]\sigma})^+. \qquad \text{(D.20)}
$$

Taking a second action of $\mathbb{Q}$, we obtain

$$
\mathbb{Q}^2(\varpi^\pm_{\mu\nu}) = (\mathbb{Q}^2\varpi)^\pm_{\mu\nu} \mp (\Psi^\sigma{}_{[\mu}(\mathbb{Q}\varpi)^+_{\nu]\sigma})^- \pm (\Psi^\sigma{}_{[\mu}(\mathbb{Q}\varpi)^-_{\nu]\sigma})^+
$$

$$\mp \mathbb{Q}\left((\Psi^{\sigma}{}_{[\mu}\varpi^{+}_{\nu]\sigma})^{-} - (\Psi^{\sigma}{}_{[\mu}\varpi^{-}_{\nu]\sigma})^{+}\right). \tag{D.21}$$

After a series of Levi-Civita contractions and spliting into self-dual and anti-self-dual components, the final term above can be computed as

$$\begin{aligned}\mathbb{Q}\left((\Psi^{\sigma}{}_{[\mu}\varpi^{+}_{\nu]\sigma})^{-} - (\Psi^{\sigma}{}_{[\mu}\varpi^{-}_{\nu]\sigma})^{+}\right) = &- (\Psi^{\sigma}{}_{[\mu}(\mathbb{Q}\varpi)^{+}_{\nu]\sigma})^{-} + (\Psi^{\sigma}{}_{[\mu}(\mathbb{Q}\varpi)^{-}_{\nu]\sigma})^{+} \\ &+ \frac{1}{2}\sqrt{g}\epsilon_{\mu\nu\alpha\beta}(\nabla^{\alpha}\Phi^{\sigma})\varpi_{\sigma}{}^{\beta} \\ &- \frac{1}{2}((\nabla_{\mu}\Phi^{\sigma})\varpi^{+}_{\sigma\nu} + (\nabla_{\nu}\Phi^{\sigma})\varpi^{+}_{\mu\sigma}) \\ &+ \frac{1}{2}((\nabla_{\mu}\Phi^{\sigma})\varpi^{-}_{\sigma\nu} + (\nabla_{\nu}\Phi^{\sigma})\varpi^{-}_{\mu\sigma}).\end{aligned} \tag{D.22}$$

We then obtain

$$\begin{aligned}\mathbb{Q}^2\varpi^{\pm}_{\mu\nu} &= (\mathbb{Q}^2\varpi)^{\pm}_{\mu\nu} \mp \frac{1}{2}\sqrt{g}\epsilon_{\mu\nu\alpha\beta}(\nabla^{\alpha}\Phi^{\sigma})\varpi_{\sigma}{}^{\beta} \\ &\qquad \pm \frac{1}{2}((\nabla_{\mu}\Phi^{\sigma})\varpi^{+}_{\sigma\nu} + (\nabla_{\nu}\Phi^{\sigma})\varpi^{+}_{\mu\sigma}) \mp \frac{1}{2}((\nabla_{\mu}\Phi^{\sigma})\varpi^{-}_{\sigma\nu} + (\nabla_{\nu}\Phi^{\sigma})\varpi^{-}_{\mu\sigma}), \\ &= (\delta_{\phi}\varpi)^{\pm}_{\mu\nu} + (\mathcal{L}^{(A)}_{\Phi}\varpi)^{\pm}_{\mu\nu} \mp \frac{1}{2}\sqrt{g}\epsilon_{\mu\nu\alpha\beta}(\nabla^{\alpha}\Phi^{\sigma})\varpi_{\sigma}{}^{\beta} \\ &\qquad \pm \frac{1}{2}((\nabla_{\mu}\Phi^{\sigma})\varpi^{+}_{\sigma\nu} + (\nabla_{\nu}\Phi^{\sigma})\varpi^{+}_{\mu\sigma}) \mp \frac{1}{2}((\nabla_{\mu}\Phi^{\sigma})\varpi^{-}_{\sigma\nu} + (\nabla_{\nu}\Phi^{\sigma})\varpi^{-}_{\mu\sigma}), \\ &= (\delta_{\phi}\varpi^{\pm})_{\mu\nu} + (\mathcal{L}^{(A)}\varpi^{\pm})_{\mu\nu}.\end{aligned} \tag{D.23}$$

We thus see that the extra terms are precisely those that change the (anti-) self-dual part of the gauge covariant Lie derivative of ϖ to the gauge covariant Lie derivative of $\varpi^{\pm}$.

E Various Computations

E.1 $(\Psi^{\sigma}{}_{[\mu}(\Psi^{\rho}{}_{[\sigma}\chi_{\nu]]\rho})^{-})^{+} = -\frac{1}{2}\Psi^{\rho\sigma}\Psi_{\rho[\mu}\chi_{\nu]\sigma}$

First, recall that

$$(\Psi^{\sigma}{}_{[\mu}\chi_{\nu]\sigma})^{-} = -\frac{1}{4}\Psi^{\sigma}{}_{\sigma}\chi_{\mu\nu} + \frac{1}{2}\sqrt{g}\epsilon_{\mu\nu\rho\sigma}\Psi^{\rho}{}_{\gamma}\chi_{\gamma\sigma} \tag{E.1}$$

so that

$$\begin{aligned}(\Psi^{\sigma}{}_{[\mu}(\Psi^{\rho}{}_{[\sigma}\chi_{\nu]]\rho})^{-})^{+} &= \left(\Psi^{\sigma}{}_{[\mu}\left(-\frac{1}{4}\Psi^{\rho}{}_{\rho}\chi_{\sigma|\nu]} + \frac{1}{2}\sqrt{g}\epsilon_{\sigma|\nu]\gamma\lambda}\Psi^{\gamma}{}_{\delta}\chi^{\delta\lambda}\right)\right)^{+} \\ &= \frac{1}{4}\Psi^{\rho}{}_{\rho}(\Psi^{\sigma}{}_{[\mu}\chi_{\nu]\sigma})^{+} + \frac{1}{2}\sqrt{g}(\epsilon_{\sigma[\nu|\gamma\lambda}\Psi^{\sigma}{}_{\mu]}\Psi^{\gamma}{}_{\delta}\chi^{\delta\lambda})^{+}\end{aligned} \tag{E.2}$$

The first term above vanishes, as

$$(\Psi^{\sigma}{}_{[\mu}\chi_{\nu]\sigma})^{+} = -\frac{1}{4}\Psi^{\sigma}{}_{\sigma}\chi_{\mu\nu}. \tag{E.3}$$

Next, we employ the self-duality of χ, and contract anti-symmetric tensors to find

$$\begin{aligned}\frac{1}{2}\sqrt{g}(\epsilon_{\sigma[\nu|\gamma\lambda}\Psi^{\sigma}{}_{\mu]}\Psi^{\gamma}{}_{\delta}\chi^{\delta\lambda})^{+} &= \frac{1}{4}\left(\epsilon_{\sigma[\nu|\lambda\eta}\epsilon^{\delta\eta\gamma\rho}\Psi^{\sigma}{}_{\mu]}\Psi^{\lambda}{}_{\delta}\chi_{\gamma\rho}\right)^{+}, \\ &= \frac{1}{2}\left((\delta_{\sigma}{}^{\delta}\delta_{[\nu}{}^{\gamma}\delta_{\lambda}{}^{\rho} - \delta_{\sigma}{}^{\gamma}\delta_{[\nu}{}^{\delta}\delta_{\lambda}{}^{\rho} + \delta_{\sigma}{}^{\gamma}\delta_{[\nu}{}^{\rho}\delta_{\lambda}{}^{\delta})\Psi^{\sigma}{}_{\mu]}\Psi^{\lambda}{}_{\mu]}\Psi^{\lambda}{}_{\delta}\chi_{\gamma\rho}\right)^{+}, \\ &= \frac{1}{2}\left(\Psi^{\sigma}{}_{[\mu}\Psi^{\rho}{}_{\sigma}\chi_{\nu]\rho} - \Psi^{\sigma}{}_{[\mu}\Psi^{\rho}{}_{\nu]}\chi_{\sigma\rho} + \Psi^{\sigma}{}_{[\mu}\Psi^{\rho}{}_{\rho}\chi_{\sigma|\nu]}\right)^{+}.\end{aligned} \tag{E.4}$$

Above, the second term vanishes due to a cycle of antisymmetric conditions and the third term vanishes for the same reason the earlier trace term vanished. This leaves us with

$$(\Psi^\sigma{}_{[\mu}(\Psi^\rho{}_{[\sigma}\chi_{\nu]]\rho})^-)^+ = -\frac{1}{2}(\Psi^{\rho\sigma}\Psi_{\sigma[\mu}\chi_{\nu]\rho})^+. \tag{E.5}$$

Here, the self-dual projection on the right hand side is redundant, as, for any anti-symmetric tensor $A_{[\mu\nu]}$, we have

$$\begin{aligned}\frac{1}{2}\sqrt{g}\epsilon_{\mu\nu\lambda\eta}A^{\sigma\lambda}\chi^\eta{}_\sigma &= \frac{1}{4}\epsilon_{\mu\nu\lambda\eta}\epsilon^{\sigma\eta\gamma\delta}A^\lambda{}_\sigma\chi_{\gamma\delta},\\ &= \frac{1}{2}(\delta^\sigma_\mu\delta^\gamma_\nu\delta^\delta_\lambda - \delta^\gamma_\mu\delta^\sigma_\nu\delta^\delta_\lambda + \delta^\gamma_\mu\delta^\delta_\nu\delta^\sigma_\lambda)A^\lambda{}_\sigma\chi_{\gamma\delta},\\ &= \frac{1}{2}A^\lambda{}_\mu\chi_{\nu\lambda} - \frac{1}{2}A^\lambda{}_\nu\chi_{\mu\lambda} + \frac{1}{2}A^\lambda{}_\lambda\chi_{\mu\nu},\\ &= A^\sigma{}_{[\mu}\chi_{\nu]\sigma}.\end{aligned} \tag{E.6}$$

So we conclude,

$$(\Psi^\sigma{}_{[\mu}(\Psi^\rho{}_{[\sigma}\chi_{\nu]]\rho})^-)^+ = -\frac{1}{2}\Psi^{\rho\sigma}\Psi_{\rho[\mu}\chi_{\nu]\sigma}. \tag{E.7}$$

E.2 $\{\widetilde{\mathsf{d}}, \mathsf{K}\}H = -\{\widetilde{\mathsf{d}}, \Delta_H\}H$

Here we provide proof of the relation (1.150). We will work rather methodically given the immense number of terms in the computation. In order to reduce the complexity, we begin by working in a presentations of our transformations that do not contain self-dual or anti-self-dual projections. Therefore we can use the various identities of Appendix D to express $\widetilde{\mathsf{d}}\chi$, $\widetilde{\mathsf{d}}H$, and $\mathsf{K}H$ explictly withouth any self-dual or anti-self-dual projection operators. We have

$$\begin{aligned}\widetilde{\mathsf{d}}\chi_{\mu\nu} &= -(\Psi^\sigma{}_{[\mu}\chi_{\nu]\sigma})^-\\ &= -\Psi^\sigma{}_{[\mu}\chi_{\nu]\sigma} - \frac{1}{4}\Psi^\sigma{}_\sigma\chi_{\mu\nu},\end{aligned} \tag{E.8}$$

$$\widetilde{\mathsf{d}} H_{\mu\nu} = -(\Psi^{\sigma}{}_{[\mu} H_{\nu]\sigma})^{-}$$
$$= -\Psi^{\sigma}{}_{[\mu} H_{\nu]\sigma} - \frac{1}{4}\Psi^{\sigma}{}_{\sigma} H_{\mu\nu}, \tag{E.9}$$

and

$$\begin{aligned}
\mathsf{K} H_{\mu\nu} &= \Phi^{\sigma} D_{\sigma}\chi_{\mu\nu} + ((\nabla_{\mu}\Phi^{\sigma})\chi_{\sigma\nu} + (\nabla_{\nu}\Phi^{\sigma})\chi_{\mu\sigma})^{+}\,, \\
&= \Phi^{\sigma} D_{\sigma}\chi_{\mu\nu} + \frac{1}{2}(\nabla_{\mu}\Phi^{\sigma})\chi_{\sigma\nu} + \frac{1}{2}(\nabla_{\nu}\Phi^{\sigma})\chi_{\mu\sigma} + \frac{1}{2}\sqrt{g}\epsilon_{\mu\nu\lambda\eta}\nabla^{\lambda}\Phi_{\sigma}\chi^{\sigma\eta} \\
&= \Phi^{\sigma} D_{\sigma}\chi_{\mu\nu} + \frac{1}{2}(\nabla_{\mu}\Phi^{\sigma})\chi_{\sigma\nu} + \frac{1}{2}(\nabla_{\nu}\Phi^{\sigma})\chi_{\mu\sigma} + \frac{1}{4}\epsilon_{\mu\nu\lambda\eta}\epsilon^{\sigma\eta\gamma\delta}\nabla^{\lambda}\Phi_{\sigma}\chi_{\gamma\delta} \\
&= \Phi^{\sigma} D_{\sigma}\chi_{\mu\nu} + \frac{1}{2}(\nabla_{\mu}\Phi^{\sigma})\chi_{\sigma\nu} + \frac{1}{2}(\nabla_{\nu}\Phi^{\sigma})\chi_{\mu\sigma} \\
&\qquad + \frac{1}{2}(\delta^{\sigma}_{\mu}\delta^{\gamma}_{\nu}\delta^{\delta}_{\lambda} - \delta^{\gamma}_{\mu}\delta^{\sigma}_{\nu}\delta^{\delta}_{\lambda} + \delta^{\gamma}_{\mu}\delta^{\delta}_{\nu}\delta^{\sigma}_{\lambda})\nabla^{\lambda}\Phi_{\sigma}\chi_{\gamma\delta} \\
&= \Phi^{\sigma} D_{\sigma}\chi_{\mu\nu} + \frac{1}{2}(\nabla_{\mu}\Phi^{\sigma})\chi_{\sigma\nu} + \frac{1}{2}(\nabla_{\nu}\Phi^{\sigma})\chi_{\mu\sigma} + \frac{1}{2}(\nabla^{\sigma}\Phi_{\mu})\chi_{\nu\sigma} \\
&\qquad - \frac{1}{2}(\nabla^{\sigma}\Phi_{\nu})\chi_{\mu\sigma} + \frac{1}{2}(\nabla_{\sigma}\Phi^{\sigma})\chi_{\mu\nu} \\
&= \Phi^{\sigma} D_{\sigma}\chi_{\mu\nu} + \frac{1}{2}(\nabla_{\sigma}\Phi^{\sigma})\chi_{\mu\nu} - \frac{1}{2}(\nabla_{\mu}\Phi^{\sigma} - \nabla^{\sigma}\Phi_{\mu})\chi_{\nu\sigma} + \frac{1}{2}(\nabla_{\nu}\Phi^{\sigma} - \nabla^{\sigma}\Phi_{\nu})\chi_{\mu\sigma}.
\end{aligned} \tag{E.10}$$

We then start with first side of the relation as

$$\{\widetilde{\mathsf{d}}, \Delta_H\} H_{\mu\nu} = \widetilde{\mathsf{d}}\left(-\frac{1}{2}\Psi^{\rho\sigma}\Psi_{\rho[\mu}\chi_{\nu]\sigma}\right) + \Delta_H\left(-\Psi^{\sigma}_{[\mu} H_{\nu]\sigma} - \frac{1}{4}\Psi^{\sigma}{}_{\sigma} H_{\mu\nu}\right) \tag{E.11}$$

$$\begin{aligned}
&= -\frac{1}{2}(\nabla^{\rho}\Phi^{\sigma} + \nabla^{\sigma}\Phi^{\rho})\Psi_{\rho[\mu}\chi_{\nu]\sigma} + \frac{1}{2}\Psi^{\rho\sigma}(\nabla_{\rho}\Phi_{[\mu} + \nabla_{[\mu}\Phi_{\rho})\chi_{\nu]\sigma} \\
&\qquad - \frac{1}{4}\Psi^{\rho\sigma}\Psi_{\rho\mu}(-\Psi^{\gamma}{}_{[\nu}\chi_{\sigma]\gamma} - \frac{1}{4}\Psi^{\gamma}{}_{\gamma}\chi_{\nu\sigma}) \\
&\qquad + \frac{1}{4}\Psi^{\rho\sigma}\Psi_{\rho\nu}(-\Psi^{\gamma}{}_{[\mu}\chi_{\sigma]\gamma} - \frac{1}{4}\Psi^{\gamma}{}_{\gamma}\chi_{\mu\sigma}) \\
&\qquad - \frac{1}{4}\Psi^{\sigma}{}_{\mu}\Psi^{\rho\gamma}\Psi_{\rho[\nu}\chi_{\sigma]\gamma} + \frac{1}{4}\Psi^{\sigma}{}_{\nu}\Psi^{\rho\gamma}\Psi_{\rho[\mu}\chi_{\sigma]\gamma} - \frac{1}{8}\Psi^{\gamma}{}_{\gamma}\Psi^{\rho\sigma}\Psi_{\rho[\mu}\chi_{\nu]\sigma} \\
&= -\frac{1}{2}(\nabla^{\rho}\Phi^{\sigma} + \nabla^{\sigma}\Phi^{\rho})\Psi_{\rho[\mu}\chi_{\nu]\sigma} + \frac{1}{2}\Psi^{\rho\sigma}(\nabla_{\rho}\Phi_{[\mu} + \nabla_{[\mu}\Phi_{\rho})\chi_{\nu]\sigma}
\end{aligned}$$

$$+\frac{1}{16}\Psi^{\gamma}{}_{\gamma}\Psi^{\rho\sigma}\Psi_{\rho\mu}\chi_{\nu\sigma}-\frac{1}{16}\Psi^{\gamma}{}_{\gamma}\Psi^{\rho\sigma}\Psi_{\rho\nu}\chi_{\mu\sigma}-\frac{1}{8}\Psi^{\gamma}{}_{\gamma}\Psi_{\rho[\mu}\chi_{\nu]\sigma} \tag{E.12}$$

$$\begin{aligned}&+\frac{1}{4}\Psi^{\rho\sigma}\Psi_{\rho\mu}\Psi^{\gamma}{}_{[\nu}\chi_{\sigma]\gamma}-\frac{1}{4}\Psi^{\rho\sigma}\Psi_{\rho\nu}\Psi^{\gamma}{}_{[\mu}\chi_{\sigma]\gamma}-\frac{1}{4}\Psi^{\sigma}{}_{\mu}\Psi^{\rho\gamma}\Psi_{\rho[\nu}\chi_{\sigma]\gamma}\\&+\frac{1}{4}\Psi^{\sigma}{}_{\nu}\Psi^{\rho\gamma}\Psi_{\rho[\mu}\chi_{\sigma]\gamma}\\=&-\frac{1}{2}(\nabla^{\rho}\Phi^{\sigma}+\nabla^{\sigma}\Phi^{\rho})\Psi_{\rho[\mu}\chi_{\nu]\sigma}+\frac{1}{2}\Psi^{\rho\sigma}(\nabla_{\rho}\Phi_{[\mu}+\nabla_{[\mu}\Phi_{\rho})\chi_{\nu]\sigma}.\end{aligned} \tag{E.13}$$

Next, we break $\{\widetilde{\mathsf{d}},\mathsf{K}\}H_{\mu\nu}$ into parts. First, we find

$$\begin{aligned}(\mathsf{K}\widetilde{\mathsf{d}})H_{\mu\nu}&=\mathsf{K}\left(-\Psi^{\sigma}_{[\mu}H_{\nu]\sigma}-\frac{1}{4}\Psi^{\sigma}{}_{\sigma}H_{\mu\nu}\right)\\&=\Psi^{\sigma}{}_{[\mu}\Bigg(\Phi^{\rho}D_{\rho}\chi_{\nu]\sigma}+\frac{1}{2}(\nabla_{\rho}\Phi^{\rho})\chi_{\nu]\sigma}-\frac{1}{2}(\nabla_{\nu]}\Phi^{\rho}-\nabla^{\rho}\Phi_{\nu]})\chi_{\sigma\rho}\\&\qquad\qquad+\frac{1}{2}(\nabla_{\sigma}\Phi^{\rho}-\nabla^{\rho}\Phi_{\sigma})\chi_{\nu]\rho}\Bigg)\\&\quad+\frac{1}{4}\Psi^{\sigma}{}_{\sigma}\Bigg(\Phi^{\rho}D_{\rho}\chi_{\mu\nu}+\frac{1}{2}(\nabla_{\rho}\Phi^{\rho})\chi_{\mu\nu}-\frac{1}{2}(\nabla_{\mu}\Phi^{\rho}-\nabla^{\rho}\Phi_{\mu})\chi_{\nu\rho}\\&\qquad\qquad+\frac{1}{2}(\nabla_{\nu}\Phi^{\rho}-\nabla^{\rho}\Phi_{\nu})\chi_{\mu\sigma}\Bigg)\end{aligned} \tag{E.14}$$

For the other direction, we compute the action of $\widetilde{\mathsf{d}}$ on each individual term of $\mathsf{K}H_{\mu\nu}$, . Additionally, for the first time, we will need to vary the metric connection. It satisfies

$$\widetilde{\mathsf{d}}\Gamma^{\rho}{}_{\mu\nu}=g^{\rho\sigma}(\nabla_{(\mu}\Psi_{\nu)\sigma}-\frac{1}{2}\nabla_{\sigma}\Psi_{\mu\nu}), \tag{E.15}$$

where we use parentheses in our indices to indicate symmetrization. This leads to

$$\widetilde{\mathsf{d}}(\nabla_{\mu}\Phi^{\sigma})=g^{\sigma\gamma}(\nabla_{(\mu}\Psi_{\rho)\gamma}-\frac{1}{2}\nabla_{\gamma}\Psi_{\mu\rho})\Phi^{\rho}, \tag{E.16}$$

$$\widetilde{\mathsf{d}}(\nabla_{\sigma}\Phi_{\mu})=\Psi_{\mu\rho}\nabla_{\sigma}\Phi^{\rho}+(\nabla_{(\sigma}\Psi_{\rho)\mu}-\frac{1}{2}\nabla_{\mu}\Psi_{\sigma\rho})\Phi^{\rho}, \tag{E.17}$$

$$\widetilde{\mathsf{d}}(\nabla_\sigma \Phi^\sigma) = \frac{1}{2}(\nabla_\rho \Psi^\sigma{}_\sigma)\Phi^\rho. \tag{E.18}$$

Term by term, we find

$$\begin{aligned}
\widetilde{\mathsf{d}}(\Phi^\sigma D_\sigma \chi_{\mu\nu}) &= \Phi^\sigma D_\sigma \left(-\Psi^\rho{}_{[\mu}\chi_{\nu]\rho} - \frac{1}{4}\Psi^\rho{}_\rho \chi_{\mu\nu} \right) - \Phi^\sigma(\widetilde{\mathsf{d}}\Gamma^\rho{}_{\sigma\mu})\chi_{\rho\nu} - \Phi^\sigma(\widetilde{\mathsf{d}}\Gamma^\rho{}_{\sigma\nu})\chi_{\mu\rho} \\
&= -\Phi^\sigma \Psi^\rho{}_{[\mu} D_\sigma \chi_{\nu]\rho} - \frac{1}{4}\Phi^\sigma \Psi^\rho{}_\rho D_\sigma \chi_{\mu\nu} - \Phi^\sigma(\nabla_\sigma \Psi^\rho{}_{[\mu})\chi_{\nu]\rho} \\
&\qquad - \frac{1}{4}\Phi^\sigma(\nabla_\sigma \Psi^\rho{}_\rho)\chi_{\mu\nu} - \Phi^\rho(g^{\rho\gamma}(\nabla_{(\sigma}\Psi_{\nu)\gamma} - \frac{1}{2}\nabla_\gamma \Psi_{\sigma\mu}))\chi_{\rho\nu} \\
&\qquad - \Phi^\sigma(g^{\rho\gamma}(\nabla_{(\sigma}\Psi_{\nu)\gamma} - \frac{1}{2}\nabla_\gamma \Psi_{\sigma\nu}))\chi_{\mu\rho} \\
&= -\Psi^\sigma{}_{[\mu}(\Phi^\rho D_\sigma \chi_{\nu]\sigma}) - \frac{1}{4}\Psi^\sigma{}_\sigma(\Phi^\rho D_\rho \chi_{\mu\nu}) \\
&\qquad - \frac{1}{4}(\nabla_\rho \Psi^\sigma{}_\sigma)\Phi^\rho \chi_{\mu\nu} - \Phi^\sigma(\nabla_\sigma \Psi^\rho{}_{[\mu})\chi_{\nu]\rho} \\
&\qquad + \Phi^\sigma(\frac{1}{2}(\nabla_\sigma \Psi_\mu{}^\rho)\chi_{\nu\rho} + \frac{1}{2}(\nabla_\mu \Psi_\sigma{}^\rho)\chi_{\nu\rho}) + \frac{1}{2}\Phi^\sigma(\nabla_\rho \Psi_{\sigma\mu})\chi^\rho{}_\nu \\
&\qquad - \Phi^\sigma(\frac{1}{2}(\nabla_\sigma \Psi_\nu{}^\rho)\chi_{\mu\rho} + \frac{1}{2}(\nabla_\nu \Psi_\sigma{}^\rho)\chi_{\mu\rho}) - \frac{1}{2}\Phi^\sigma(\nabla_\rho \Psi_{\sigma\nu})\chi^\rho{}_\mu \\
&= -\Psi^\sigma{}_{[\mu}(\Phi^\rho D_\sigma \chi_{\nu]\sigma}) - \frac{1}{4}\Psi^\sigma{}_\sigma(\Phi^\rho D_\rho \chi_{\mu\nu}) - \frac{1}{4}(\nabla_\rho \Psi^\sigma{}_\sigma)\Phi^\rho \chi_{\mu\nu} \\
&\qquad + (\nabla_{[\mu}\Psi_{\sigma\rho})\Phi^\sigma \chi_{\nu]}{}^\rho - (\nabla_\rho \Psi_{\sigma[\mu}\Phi^\sigma \chi_{\nu]}{}^\rho,
\end{aligned} \tag{E.19}$$

$$\begin{aligned}
\widetilde{\mathsf{d}}\left(\frac{1}{2}(\nabla_\sigma \Phi^\sigma)\chi_{\mu\nu}\right) &= \frac{1}{4}(\nabla_\rho \Psi^\sigma{}_\sigma)\Phi^\rho \chi_{\mu\nu} + \frac{1}{2}(\nabla_\sigma \Phi^\sigma)(-\Psi^\rho_{[\mu}\chi_{\nu]\rho} - \frac{1}{4}\Psi^\rho{}_\rho \chi_{\mu\nu}) \\
&= \frac{1}{4}(\nabla_\rho \Psi^\sigma{}_\sigma)\Phi^\rho \chi_{\mu\nu} - \frac{1}{2}\Psi^\sigma{}_{[\mu}(\nabla_\rho \Phi^\rho)\chi_{\nu]\sigma} - \frac{1}{4}\Psi^\sigma{}_\sigma(\nabla_\rho \Phi^\rho)\chi_{\mu\nu},
\end{aligned} \tag{E.20}$$

$$\widetilde{\mathsf{d}}\left(-\frac{1}{2}(\nabla_\mu\Phi^\sigma)\chi_{\nu\sigma}\right) = -\frac{1}{2}(g^{\sigma\gamma}(\nabla_{(\nu}\Psi_{\rho)\gamma} - \frac{1}{2}\nabla_\gamma\Psi_{\mu\rho})\Phi^\rho\chi_{\nu\sigma})$$
$$- \frac{1}{2}(\nabla_\mu\Phi^\sigma)(-\Psi^\gamma{}_{[\nu}\chi_{\sigma]\gamma} - \frac{1}{4}\Psi^\gamma{}_\gamma\chi_{\nu\sigma})$$
$$= -\frac{1}{4}(\nabla_\mu\Psi_{\sigma\rho})\Phi^\sigma\chi_\nu{}^\rho - \frac{1}{4}(\nabla_\sigma\Psi_{\mu\rho})\Phi^\sigma\chi_\nu{}^\rho + \frac{1}{4}(\nabla_\rho\Psi_{\mu\sigma})\Phi^\sigma\chi_\nu{}^\rho$$
$$+ \frac{1}{2}\Psi^\sigma{}_{[\nu}(\nabla_\mu\Phi^\rho)\chi_{\rho]\sigma} + \frac{1}{8}\Psi^\sigma{}_\sigma(\nabla_\mu\Phi^\rho)\chi_{\nu\rho}, \tag{E.21}$$

$$\widetilde{\mathsf{d}}\left(\frac{1}{2}(\nabla_\nu\Phi^\sigma)\chi_{\mu\sigma}\right) = \frac{1}{4}(\nabla_\nu\Psi_{\sigma\rho})\Phi^\sigma\chi_\mu{}^\rho + \frac{1}{4}(\nabla_\sigma\Psi_{\nu\rho})\Phi^\sigma\chi_\mu{}^\rho - \frac{1}{4}(\nabla_\rho\Psi_{\nu\sigma})\Phi^\sigma\chi_\mu{}^\rho$$
$$- \frac{1}{2}\Psi^\sigma{}_{[\mu}(\nabla_\nu\Phi^\rho)\chi_{\rho]\sigma} - \frac{1}{8}\Psi^\sigma{}_\sigma(\nabla_\nu\Phi^\rho)\chi_{\mu\rho}, \tag{E.22}$$

$$\widetilde{\mathsf{d}}\left(\frac{1}{2}(\nabla_\sigma\Phi_\mu)\chi_\nu{}^\sigma\right) = \frac{1}{2}(\Psi_{\mu\rho}\nabla_\sigma\Phi^\rho + (\nabla_{(\sigma}\Psi_{\rho)\mu} - \frac{1}{2}\nabla_\mu\Psi_{\sigma\rho})\Phi^\rho)\chi_\nu{}^\sigma - \frac{1}{2}\Psi^{\sigma\rho}(\nabla_\sigma\Phi_\mu)\chi_{\nu\rho}$$
$$+ \frac{1}{2}(\nabla_\sigma\Phi_\mu)g^{\sigma\rho}(-\Psi^\gamma{}_{[\nu}\chi_{\rho]\gamma} - \frac{1}{4}\Psi^\gamma{}_\gamma\chi_{\nu\rho})$$
$$= \frac{1}{2}\Psi^\sigma{}_\mu(\nabla^\rho\Phi_\sigma)\chi_{\nu\sigma} - \frac{1}{4}\Psi^{\sigma\rho}(\nabla_\sigma\Phi_\mu)\chi_{\nu\rho} - \frac{1}{4}\Psi^\sigma{}_\nu(\nabla_\rho\Phi_\mu)\chi^\rho{}_\sigma$$
$$- \frac{1}{8}\Psi^\sigma{}_\sigma(\nabla^\rho\Phi_\mu)\chi_{\nu\rho} + \frac{1}{4}(\nabla_\rho\Psi_{\sigma\mu})\Phi^\sigma\chi_\nu{}^\rho + \frac{1}{4}(\nabla_\sigma\Psi_{\rho\mu})\Phi^\sigma\chi_\nu{}^\rho$$
$$- \frac{1}{4}(\nabla_\mu\Psi_{\sigma\rho})\Phi^\sigma\chi_\nu{}^\rho. \tag{E.23}$$

$$\widetilde{\mathsf{d}}\left(-\frac{1}{2}(\nabla_\sigma\Phi_\nu)\chi_\mu{}^\sigma\right) = -\frac{1}{2}\Psi^\sigma{}_\nu(\nabla^\rho\Phi_\sigma)\chi_{\mu\sigma} + \frac{1}{4}\Psi^{\sigma\rho}(\nabla_\sigma\Phi_\nu)\chi_{\mu\rho}$$
$$+ \frac{1}{4}\Psi^\sigma{}_\mu(\nabla_\rho\Phi_\nu)\chi^\rho{}_\sigma + \frac{1}{8}\Psi^\sigma{}_\sigma(\nabla^\rho\Phi_\nu)\chi_{\mu\rho}$$
$$- \frac{1}{4}(\nabla_\rho\Psi_{\sigma\nu})\Phi^\sigma\chi_\mu{}^\rho - \frac{1}{4}(\nabla_\sigma\Psi_{\rho\nu})\Phi^\sigma\chi_\mu{}^\rho$$
$$+ \frac{1}{4}(\nabla_\nu\Psi_{\sigma\rho})\Phi^\sigma\chi_\mu{}^\rho. \tag{E.24}$$

Collecting (E.21)-(E.24), we find

$$\begin{aligned}\widetilde{\mathsf{d}}\left(-(\nabla_{[\mu}\Phi^{\sigma})\chi_{\nu]\sigma}+(\nabla_{\sigma}\Phi_{[\mu})\chi_{\nu]}{}^{\sigma}\right) = &-(\nabla_{[\mu}\Psi_{\sigma\rho})\Phi^{\sigma}\chi_{\nu]}{}^{\rho}+(\nabla_{\rho}\Psi_{\sigma[\mu})\Phi^{\sigma}\chi_{\nu]}{}^{\rho}\\ &-\frac{1}{4}\Psi^{\sigma}{}_{\sigma}(-\frac{1}{2}(\nabla_{\mu}\Phi^{\rho}-\nabla^{\rho}\Phi_{\mu})\chi_{\nu\rho}\\ &+\frac{1}{2}(\nabla_{\nu}\Phi^{\rho}-\nabla^{\rho}\Phi_{\nu})\chi_{\mu\sigma})+\frac{1}{2}\Psi^{\sigma}{}_{[\mu}(\nabla_{\nu]}\Phi^{\rho})\chi_{\sigma\rho}\\ &-\frac{1}{2}\Psi^{\sigma\rho}(\nabla_{[\mu}\Phi_{\rho})\chi_{\nu]\sigma}+\Psi^{\sigma}{}_{[\mu}(\nabla^{\rho}\Phi_{\sigma})\chi_{\nu]\rho}\\ &-\frac{1}{2}\Psi^{\sigma}{}_{[\mu}(\nabla^{\rho}\Phi_{\nu]})\chi_{\sigma\rho}-\frac{1}{2}\Psi^{\sigma\rho}(\nabla_{\sigma}\Phi_{[\mu})\chi_{\nu]\rho}.\end{aligned} \tag{E.25}$$

All the pieces together gives

$$\begin{aligned}(\widetilde{\mathsf{d}}\mathsf{K})H_{\mu\nu} = &-\Psi^{\sigma}{}_{[\mu}(\Phi^{\rho}D_{\rho}\chi_{\nu]\sigma})-\frac{1}{4}\Psi^{\sigma}{}_{\sigma}(\Phi^{\rho}D_{\rho}\chi_{\mu\nu})-\frac{1}{2}\Psi^{\sigma}{}_{[\mu}(\nabla_{\rho}\Phi^{\rho})\chi_{\nu]\sigma}\\ &-\frac{1}{4}\Psi^{\sigma}{}_{\sigma}((\nabla_{\rho}\Phi^{\rho})\chi_{\mu\nu}-\frac{1}{2}(\nabla_{\mu}\Phi^{\rho}-\nabla^{\rho}\Phi_{\mu})\chi_{\nu\rho}+\frac{1}{2}(\nabla_{\nu}\Phi^{\rho}-\nabla^{\rho}\Phi_{\nu})\chi_{\mu\sigma})\\ &+\Psi^{\sigma}{}_{[\mu}(\frac{1}{2}(\nabla_{\nu]}\Phi^{\rho})\chi_{\sigma\rho}+(\nabla^{\rho}\Phi_{\sigma})\chi_{\nu]\rho}-\frac{1}{2}(\nabla^{\rho}\Phi_{\nu]})\chi_{\sigma\rho})\\ &-\frac{1}{2}\Psi^{\sigma\rho}((\nabla_{[\mu}\Phi_{\rho})+(\nabla_{\rho}\Phi_{[\mu}))\chi_{\nu]\sigma}.\end{aligned} \tag{E.26}$$

This finally gives us

$$\{\widetilde{\mathsf{d}},\mathsf{K}\} = -\frac{1}{2}\Psi^{\sigma\rho}(\nabla_{[\mu}\Phi_{\rho}+\nabla_{\rho}\Phi_{[\mu})\chi_{\nu]\sigma}+\frac{1}{2}\Psi^{\sigma}{}_{[\mu}(\nabla^{\rho}\Phi_{\sigma}+\nabla_{\sigma}\Phi^{\rho})\chi_{\nu]\rho}. \tag{E.27}$$

Thus, comparing this to (E.13), we see that, indeed

$$\{\widetilde{\mathsf{d}},\mathsf{K}\}H = -\{\widetilde{\mathsf{d}},\Delta_{H}\}H. \tag{E.28}$$

F Exact + Non-exact Splitting of $\sqrt{g}\mathsf{C}_{\mathbf{IR}-}$

For the purposes of understanding the action of superconformal tensor calculus, we can write the unbarred half the $\mathbb{S}^{\mathrm{t}}$ as a entirely non-exact part plus the exact part. This is given by

$$\mathbb{S}^{\mathrm{t}}\Big|_{\text{anti-chiral}} = \mathbb{Q}(\mathbb{V} + \mathbb{A}) + \mathsf{C}, \tag{F.1}$$

where is $\mathbb{V}$ of (2.7) $\mathbb{A}$ of (2.70) and C is (2.3.2). Explicitly written in the "t" fields, we have

$$\begin{aligned}\mathbb{V} = \int_{\mathbb{X}} d^4x\sqrt{g}\,\Big[&-\frac{1}{2}\tau(F^{\mathrm{t},+}_{\mu\nu} + D^{\mathrm{t}}_{\mu\nu})\chi^{\mu\nu} - 2\tau\psi^{\mathrm{t}}_{\mu}\nabla^{\mu}\overline{a} + \frac{\partial\tau}{\partial a}\psi^{\mathrm{t}}_{\mu}\psi^{\mathrm{t}}_{\nu}\chi^{\mu\nu} - \tau(\nabla_{[\mu}\Phi_{\nu]})\overline{a}\chi^{\mu\nu} \\ &-2\tau\Phi^{\mu}\chi_{\mu\nu}\nabla^{\nu}\overline{a} + 2\frac{\partial\tau}{\partial a}\Phi^{\sigma}\psi^{\mathrm{t}}_{\mu}\chi_{\sigma\nu}\chi^{\mu\nu} + \frac{\partial\tau}{\partial a}\Phi^{\rho}\Phi^{\sigma}\chi_{\sigma\mu}\chi_{\rho\nu}\chi^{\mu\nu}\Big]\end{aligned} \tag{F.2}$$

$$\mathbb{A} = \int_{\mathbb{X}} d^4x\sqrt{g}\,\Big[2\tau\Phi^{\mu}\chi_{\mu\nu}\nabla^{\nu}\overline{a} + \frac{\partial\tau}{\partial a}\Phi^{\rho}\chi_{\rho\sigma}\chi^{\sigma\mu}\psi^{\mathrm{t}}_{\mu} - \frac{2}{3}\frac{\partial\tau}{\partial a}\Phi^{\rho}\Phi^{\sigma}\chi_{\rho\mu}\chi_{\sigma\nu}\chi^{\mu\nu}\Big] \tag{F.3}$$

and

$$\begin{aligned}\mathsf{C} = \int_{\mathbb{X}} d^4x\,\Big[&\frac{\tau}{4}\epsilon^{\mu\nu\alpha\beta}F^{\mathrm{t}}_{\mu\nu}F^{\mathrm{t}}_{\alpha\beta} - \frac{1}{2}\frac{\partial\tau}{\partial a}\epsilon^{\mu\nu\alpha\beta}\psi^{\mathrm{t}}_{\mu}\psi^{\mathrm{t}}_{\nu}F^{\mathrm{t}}_{\alpha\beta} + \frac{1}{12}\frac{\partial^2\tau}{\partial a^2}\epsilon^{\mu\nu\alpha\beta}\psi^{\mathrm{t}}_{\mu}\psi^{\mathrm{t}}_{\nu}\psi^{\mathrm{t}}_{\alpha}\psi^{\mathrm{t}}_{\beta} \\ &+\frac{\tau}{2}\epsilon^{\mu\nu\alpha\beta}F^{\mathrm{t}}_{\mu\nu}\nabla_{[\alpha}(\Phi_{\beta]}\overline{a}) + \tau\epsilon^{\mu\nu\alpha\beta}\nabla_{[\mu}(\Phi_{\nu]}\overline{a})\nabla_{[\alpha}(\Phi_{\beta]}\overline{a}) \\ &-\frac{\partial\tau}{\partial a}\epsilon^{\mu\nu\alpha\beta}\Phi^{\sigma}\chi_{\sigma\mu}\psi^{\mathrm{t}}_{\nu}F^{\mathrm{t}}_{\alpha\beta} - 2\frac{\partial\tau}{\partial a}\epsilon^{\mu\nu\alpha\beta}\Phi^{\sigma}\chi_{\sigma\mu}\psi^{\mathrm{t}}_{\nu}\nabla_{[\alpha}(\Phi_{\beta]}\overline{a}) \\ &-\frac{1}{2}\frac{\partial\tau}{\partial a}\epsilon^{\mu\nu\alpha\beta}\Phi^{\rho}\Phi^{\sigma}\chi_{\rho\mu}\chi_{\sigma\nu}F^{\mathrm{t}}_{\alpha\beta} + \frac{\partial\tau}{\partial a}\epsilon^{\mu\nu\alpha\beta}\Phi^{\rho}\Phi^{\sigma}\chi_{\rho\mu}\chi_{\sigma\nu}\nabla_{[\alpha}(\Phi_{\beta]}\overline{a}) \\ &+\frac{1}{3}\frac{\partial^2\tau}{\partial a^2}\epsilon^{\mu\nu\alpha\beta}\Phi^{\sigma}\chi_{\sigma\mu}\psi^{\mathrm{t}}_{\mu}\psi^{\mathrm{t}}_{\alpha}\psi^{\mathrm{t}}_{\beta} + \frac{1}{2}\frac{\partial^2\tau}{\partial a^2}\epsilon^{\mu\nu\alpha\beta}\Phi^{\sigma}\chi_{\sigma\mu}\Phi^{\rho}\chi_{\rho\nu}\psi^{\mathrm{t}}_{\alpha}\psi^{\mathrm{t}}_{\beta} \\ &+\frac{1}{3}\frac{\partial^2\tau}{\partial a^2}\epsilon^{\mu\nu\alpha\beta}\Phi^{\sigma}\chi_{\sigma\mu}\Phi^{\rho}\chi_{\rho\nu}\Phi^{\gamma}\chi_{\gamma\alpha}\psi^{\mathrm{t}}_{\beta}\Big]\end{aligned} \tag{F.4}$$

There are simplifications upon adding $\mathbb{V}$ and $\mathbb{A}$. We find

$$\mathbb{V}+\mathbb{A}=\int_{\mathbb{X}} d^4x\sqrt{g}\left[-\frac{1}{2}\tau(F^{\mathrm{t},+}_{\mu\nu}+D^{\mathrm{t}}_{\mu\nu})\chi^{\mu\nu}-2\tau\psi^{\mathrm{t}}_{\mu}\nabla^{\mu}\overline{a}+\frac{\partial\tau}{\partial a}\psi^{\mathrm{t}}_{\mu}\psi^{\mathrm{t}}_{\nu}\chi^{\mu\nu}\right.$$
$$\left.-\tau(\nabla_{[\mu}\Phi_{\nu]})\overline{a}\chi^{\mu\nu}-\frac{\partial\tau}{\partial a}\Phi^{\rho}\chi_{\rho\sigma}\chi^{\sigma\mu}\psi^{\mathrm{t}}_{\mu}+\frac{1}{3}\frac{\partial\tau}{\partial a}\Phi^{\rho}\Phi^{\sigma}\chi_{\rho\mu}\chi_{\sigma\nu}\chi^{\mu\nu}\right] \tag{F.5}$$

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